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ON θ-CENTRALIZERS OF SEMIPRIME RINGS

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Abstract

The main result of the present article is the following: Let R be a 2-torsion-free semiprime ring, θ be an endomorphism of R and $T\colon R \to R$ be an additive mapping such that $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all $x, y \in R$. Then T is a $\theta-$centralizer of R.

1 Introduction

This note has been motivated by the works of J. Vukman [4] and E. Albaş [1]. Throughout, R will represent an associative ring with center $Z(R)$, not necessatily with an identity element. A ring R is 2-torsion-free, if $2x = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ for $x, y \in R$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$, for $x, y \in R$. Recall that R is semiprime if $aRa = (0)$ implies $a = 0$, for every $a \in R$.

B. Zalar [5] introduced the following notion. Let R be a semiprime ring. A *left* (resp. *right*) *centralizer* of R is an additive mapping $T\colon R \to R$ satisfying $T(xy) = T(x)y$ (resp.

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$T(xy) = xT(y)$) for all $x, y \in R$. If T is a left and a right centralizer then T is a *centralizer*. In case R has an identity element, $T\colon R \to R$ is a left (resp. right) centralizer if and only if T is of the form $T(x) = ax$ (resp. $T(x) = xa$) for some fixed element $a \in R$. An additive mapping $T\colon R \to R$ is called a *left* (resp. *right*) *Jordan centralizer* in case $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for $x \in R$, and is called a *Jordan centralizer* if T satisfies $T(xy+yx) = T(x)y+yT(x) = T(y)x+xT(y)$ for all $x, y \in R$. In [5], it was shown that a Jordan centralizer of a semiprime ring is a left centralizer, and each Jordan centralizer is a centralizer.

Following ideas from M. Brešar [2], B. Zalar [5] has proved that any left (right) Jordan centralizer on a 2-torsion-free semiprime ring is a left (right) centralizer. If $T\colon R \to R$ is a centralizer, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = xT(y)x, \ \forall\, x, y \in R. \tag{1}$$

It seems natural to ask whether the converse is true. More precisely, asking for whether an additive mapping T on a ring R satisfying relation (1) is a centralizer. In [4], J. Vukman proved that the answer is affirmative in case R is a 2-torsion-free semiprime ring. The proof of his result is rather long, but it is elementary in the sense that it requires no specific knowledge concerning semiprime ring theory in order to follow the proof.

Recently, E. Albaş [1] introduced the following definitions, which are generalizations of the definitions of centralizer and Jordan centralizer. Let R be a semiprime 2-torsion-free ring, and let θ be an endomorphism of R. A *Jordan θ-centralizer* of R is an additive mapping $f\colon R \to R$ satisfying $f(xy+yx) = f(x)\theta(y)+\theta(y)f(x) = f(y)\theta(x)+\theta(x)f(y)$ for all $x, y \in R$. An additive mapping $f\colon R \to R$ is called a *left* (resp. *right*) *θ-centralizer* of R if $f(xy) = f(x)\theta(y)$ (resp. $f(xy) = \theta(x)f(y)$) for all $x, y \in R$. If f is a left and right θ-centralizer then it is natural to call f a *θ-centralizer*. It is clear that for an additive mapping $T\colon R \to R$ associated with a homomorphism $\theta\colon R \to R$, if $L_a(x) = a\theta(x)$ and $R_a(x) = \theta(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left $\theta-$centralizer andR_a is a right $\theta-$centralizer. Clearly every centralizer is a special case of a θ-centralizer with $\theta = id_R$.

An additive mapping $f\colon R \to R$ is called a *left* (resp. *right*) *Jordan θ-centralizer* of R if $f(x^2) = f(x)\theta(x)$ (resp. $f(x^2) = \theta(x)f(x)$) for all $x \in R$. It is clear that a left θ-centralizer of R is a left Jordan θ-centralizer and, analogously, a θ-centralizer of R is a Jordan θ-centralizer of R. The converse is no longer true, in general. In [1], E. Albaş proved, under some conditions, that in a 2-torsion-free semiprime ring R, every Jordan θ-centralizer is a θ-centralizer. In [3], W. Cortes and C. Haetinger proved this question changing the semiprimality condition on R. The main result of this paper is the following: Let R be a 2-torsion-free ring which has a commutator right (resp. left) nonzero divisor and let $G\colon R \to R$ be a left (resp. right) Jordan σ-centralizer mapping of R, where σ is an automorphism of R. Then G is a left (resp. right) σ-centralizer mapping of R.

Now, if $T\colon R \to R$ is a θ-centralizer associated with a function $\theta\colon R \to R$, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = \theta(x)T(y)\theta(x) \quad \forall\, x, y \in R. \tag{2}$$

Again, as J. Vukman [4] did on the centralizer case, we are asking whether an additive mapping T on a ring R satisfying relation (2) is a θ-centralizer for every $x, y \in R$. It is the aim in this paper to prove that the answer is affirmative in case R is a 2-torsion-free semiprime ring with some conditions on θ.

Otherwise unless stated, R will be a 2-torsion-free semiprime rings, and θ an endomorphism of R.

2 Results

The main goal of this paper is to prove the following

Theorem 2.1 *Let R be a 2-torsion-free semiprime ring and let $T\colon R \to R$ be an additive mapping such that $T(xyx) = \theta(x)T(y)\theta(x)$ holds for all pairs $x, y \in R$, where θ is a nonzero surjective endomorphism on R with $\theta(Z(R)) = Z(R)$. Then T is a θ-centralizer.*

Note that if we put $y = x$ in relation (2) it gives

$$T(x^3) = \theta(x)T(x)\theta(x), \quad \forall\, x \in R. \tag{3}$$

The question arises whether in a 2-torsion-free semiprime ring the above relation implies that T is a $\theta-$centralizer.

We shall prove that the answer is affirmative in case R has an identity element.

Theorem 2.2 *Let R be a 2-torsion-free semiprime ring with an identity element, θ a nonzero surjective homomorphism on R, and let $T\colon R \to R$ be an additive mapping such that $T(x^3) = \theta(x)T(x)\theta(x)$ holds for all $x \in R$. Then T is a θ-centralizer.*

3 Proofs

For the proof of Theorem 2.1 the following lemma will be needed.

Lemma 3.1 [4, Lemma 1] *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

Proof of Theorem 2.1. To prove that T is a θ-centralizer of R, we intend to prove the relation

$$[T(x), \theta(x)] = 0,\ \forall x \in R. \tag{4}$$

For the proof of the above relation we shall need the weaker relation below

$$[[T(x), \theta(x)], \theta(x)] = 0,\ \forall x \in R. \tag{5}$$

Replacing x by $x + z$ in (2), we get

$$T(xyz + zyx) = \theta(x)T(y)\theta(z) + \theta(z)T(y)\theta(x), \qquad \forall x, y, z \in R. \tag{6}$$

Putting $y = x$ and $z = y$ in (6) one obtain

$$T(x^2y + yx^2) = \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x), \quad \forall\, x, y \in R. \tag{7}$$

For $z = x^3$, relation (6) reduces to

$$T(xyx^3 + x^3yx) = \theta(x)T(y)\theta(x^3) + \theta(x^3)T(y)\theta(x), \quad \forall\, x, y \in R. \tag{8}$$

Now replace y by xyx in (7). We get

$$T(xyx^3 + x^3yx) = \theta(x)T(x)\theta(xyx) + \theta(xyx)T(x)\theta(x), \quad \forall\, x, y \in R. \tag{9}$$

The substitution $x^2y + yx^2$ for y in relation (2) gives

$$T(xyx^3 + x^3yx) = \theta(x)T(x^2y + yx^2)\theta(x),\ \forall\, x, y \in R.$$

Which implies, because of (7),

$$T(x^3yx + xyx^3) = \theta(x^2)T(x)\theta(yx) + \theta(xy)T(x)\theta(x^2), \quad \forall\, x, y \in R. \tag{10}$$

Combining (9) with (10) we arrive at

$$\theta(x)[T(x), \theta(x)]\theta(yx) - \theta(xy)[T(x), \theta(x)]\theta(x) = 0, \quad \forall\, x, y \in R. \tag{11}$$

Putting in equation (11), $a = \theta(x)[T(x), \theta(x)]$, $b = \theta(x)$, $c = -[T(x), \theta(x)]\theta(x)$ and $z = \theta(y)$, this expression can be rewritten on the form $azb + bzc = 0$, for every $z \in R$. Applying Lemma 3.1 on the above relation it follows that

$$[[T(x), \theta(x)], \theta(x)]\theta(yx) = 0, \quad \forall\, x, y \in R. \tag{12}$$

Let $\theta(y)$ be $\theta(y)[T(x), \theta(x)]$ in (12). We have

$$[[T(x), \theta(x)], \theta(x)]\theta(y)[T(x), \theta(x)]\theta(x) = 0, \quad \forall\, x, y \in R. \tag{13}$$

Right multiplication of (12) by $[T(x), \theta(x)]$ gives

$$[[T(x), \theta(x)], \theta(x)]\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{14}$$

Subtracting (14) from (13) we obtain

$$[[T(x), \theta(x)], \theta(x)]\theta(y)[[T(x), \theta(x)], \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{15}$$

Since R is semiprime and θ is onto we get, $[[T(x), \theta(x)], \theta(x)] = 0$, for all $x \in R$.

The next step is to prove the relation

$$\theta(x)[T(x), \theta(x)]\theta(x) = 0, \quad \forall\, x \in R. \tag{16}$$

Substituting x by $x + y$ in (5) we have, for every $x, y \in R$, $[[T(x), \theta(x)], \theta(y)] + [[T(x), \theta(y)], \theta(x)] + [[T(y), \theta(y)], \theta(x)] + [[T(y), \theta(x)], \theta(y)] + [[T(y), \theta(x)], \theta(x)] + [[T(x), \theta(y)], \theta(y)] = 0$. Putting $-x$ for x in the above relation and comparing the expression so obtained with the above one we get for every $x, y \in R$

$$[[T(x), \theta(x)], \theta(y)] + [[T(x), \theta(y)], \theta(x)] + [[T(y), \theta(x)], \theta(x)] = 0. \tag{17}$$

Replacing y by xyx in (17) and using (2), (5) and (17) we obtain

$$\begin{aligned} 0 &= [[T(x),\theta(x)],\theta(xyx)] + [[T(x),\theta(xyx)],\theta(x)]+\\ &\quad +[[\theta(x)T(y)\theta(x),\theta(x)],\theta(x)] =\\ &= \theta(x)[[T(x),\theta(x)],\theta(y)]\theta(x)+\\ &\quad +[[T(x),\theta(x)]\theta(yx)+\theta(x)[T(x),\theta(y)]\theta(x)+\theta(xy)[T(x),\theta(x)],\theta(x)]+\\ &\quad +[\theta(x)[T(y),\theta(x)]\theta(x),\theta(x)] =\\ &= \theta(x)[[T(x),\theta(x)],\theta(y)]\theta(x)+[T(x),\theta(x)][\theta(y),\theta(x)]\theta(x)+\\ &\quad +\theta(x)[[T(x),\theta(y)],\theta(x)]\theta(x)+\theta(x)[\theta(y),\theta(x)][T(x),\theta(x)]+\\ &\quad +\theta(x)[[T(y),\theta(x)],\theta(x)]\theta(x) =\\ &= [T(x),\theta(x)][\theta(y),\theta(x)]\theta(x)+\theta(x)[\theta(y),\theta(x)][T(x),\theta(x)] =\\ &= [T(x),\theta(x)]\theta(yx^2)-\theta(x^2y)[T(x),\theta(x)]+\\ &\quad +\theta(xyx)[T(x),\theta(x)]-[T(x),\theta(x)]\theta(xyx). \end{aligned}$$

Therefore, for every $x, y \in R$, we have

$$[T(x),\theta(x)]\theta(yx^2)-\theta(x^2y)[T(x),\theta(x)]+\theta(xyx)[T(x),\theta(x)]-[T(x),\theta(x)]\theta(xyx)=0.$$

Which reduces because of (5) and (11) to

$$[T(x),\theta(x)]\theta(yx^2)-\theta(x^2y)[T(x),\theta(x)]=0,\ \forall\, x,y\in R.$$

Left multiplication of the above relation by $\theta(x)$ gives

$$\theta(x)[T(x),\theta(x)]\theta(yx^2)-\theta(x^3y)[T(x),\theta(x)]=0,\ \forall\, x,y\in R.$$

One can replace in the above relation, according to (11), $\theta(x)[T(x),\theta(x)]\theta(yx)$ by $\theta(xy)[T(x),\theta(x)]\theta(x)$, which gives

$$\theta(xy)[T(x),\theta(x)]\theta(x^2)-\theta(x^3y)[T(x),\theta(x)]=0,\quad \forall\, x,y\in R. \tag{18}$$

Left multiplication of the above relation by $T(x)$ gives

$$T(x)\theta(xy)[T(x),\theta(x)]\theta(x^2)-T(x)\theta(x^3y)[T(x),\theta(x)]=0,\quad \forall\, x,y\in R. \tag{19}$$

Substitute $T(x)\theta(y)$ for $\theta(y)$ in (18) which leads to

$$\theta(x)T(x)\theta(y)[T(x),\theta(x)]\theta(x^2)-\theta(x^3)T(x)\theta(y)[T(x),\theta(x)]=0,\quad \forall\, x,y\in R. \tag{20}$$

Subtracting (20) from (19) we obtain for all $x, y \in R$

$$[T(x),\theta(x)]\theta(y)[T(x),\theta(x)]\theta(x^2)-[T(x),\theta(x^3)]\theta(y)[T(x),\theta(x)]=0. \tag{21}$$

Which can be rewritten in the form

$$[T(x),\theta(x^3)]\theta(y)[T(x),\theta(x)]-[T(x),\theta(x)]\theta(y)[T(x),\theta(x)]\theta(x^2)=0,\ \forall\, x,y\in R.$$

If we take $a=[T(x),\theta(x^3)]$, $b=[T(x),\theta(x)]$, $c=-[T(x),\theta(x)]\theta(x^2)$ and $z=\theta(y)$ in the above relation, it can be rewritten in the form $azb+bzc=0$, for every $z\in R$. Applying Lemma 3.1 again, it follows that

$$([T(x),\theta(x^3)]-[T(x),\theta(x)]\theta(x^2))\theta(y)[T(x),\theta(x)]=0,\quad \forall\, x,y\in R. \tag{22}$$

Which reduces for every $x, y \in R$ to

$$(\theta(x)[T(x), \theta(x)]\theta(x) + \theta(x^2)[T(x), \theta(x)])\theta(y)[T(x), \theta(x)] = 0. \tag{23}$$

Relation (5) makes it possible now to write $[T(x), \theta(x)]\theta(x)$ instead of $\theta(x)[T(x), \theta(x)]$, which means that, in the above expression, $\theta(x^2)[T(x), \theta(x)]$ can be replaced by $\theta(x)[T(x), \theta(x)]\theta(x)$. Thus we have, for every $x, y \in R$,

$$\theta(x)[T(x), \theta(x)]\theta(xy)[T(x), \theta(x)] = 0.$$

Right multiplication of the above relation by $\theta(x)$ and substituting $\theta(yx)$ for $\theta(y)$ gives finally $\theta(x)[T(x), \theta(x)]\theta(xyx)[T(x), \theta(x)]\theta(x) = 0$, for every x, y belonging to R. By the the semiprimeness of R and the surjectivity of θ we have that $\theta(x)[T(x), \theta(x)]\theta(x) = 0$ holds for every $x \in R$, and so (16) follows.

Next we prove the following relation

$$\theta(x)[T(x), \theta(x)] = 0, \quad \forall\, x \in R. \tag{24}$$

The substitution of yx for y in (11) gives, because of (16),

$$\theta(x)[T(x), \theta(x)]\theta(yx^2) = 0, \quad \forall\, x, y \in R. \tag{25}$$

Putting $\theta(y)T(x)$ for $\theta(y)$ in the above relation we obtain

$$\theta(x)[T(x), \theta(x)]\theta(y)T(x)\theta(x^2) = 0, \quad \forall\, x, y \in R. \tag{26}$$

Right multiplication of (25) by $T(x)$ gives

$$\theta(x)[T(x), \theta(x)]\theta(yx^2)T(x) = 0, \quad \forall\, x, y \in R. \tag{27}$$

Subtracting (27) from (26) we obtain $\theta(x)[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] = 0$, for every $x, y \in R$, which can be rewritten in the form

$$\theta(x)[T(x), \theta(x)]\theta(y)([T(x), \theta(x)]\theta(x) + \theta(x)[T(x), \theta(x)]) = 0,\ \forall\, x, y \in R.$$

According to (5) we can replace $[T(x), \theta(x)]\theta(x)$ in the relation above by $\theta(x)[T(x), \theta(x)]$, which gives $\theta(x)[T(x), \theta(x)]\theta(yx)[T(x), \theta(x)] = 0$, for all $x, y \in R$. So, by the surjectivity of θ and the semiprimeness of R we get $\theta(x)[T(x), \theta(x)] = 0$, for each $x \in R$. Whence relation (24) holds. It follows from (5) and (24) that

$$[T(x), \theta(x)]\theta(x) = 0,\ \forall\, x \in R.$$

Substituting x by $x + y$ in the expression above, we obtain for all $x, y \in R$ that

$$[T(x), \theta(x)]\theta(y) + [T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(y) +$$

$$[T(y), \theta(x)]\theta(x) + [T(y), \theta(x)]\theta(y) + [T(y), \theta(y)]\theta(x) = 0.$$

Replacing now x by $-x$ in this equation and comparing the relation so obtained with the above one we arrive at. $[T(x), \theta(x)]\theta(y) + [T(x), \theta(y)]\theta(x) + [T(y), \theta(x)])\theta(x) = 0$, for every $x, y \in R$.

Right multiplication of the last expression by $[T(x),\theta(x)]$ gives, because of (24), $[T(x),\theta(x)]\theta(y)[T(x),\theta(x)] = 0$, for all $x, y \in R$. So, by the surjectivity of θ and the semiprimeness of R we get (4).

Let now $A(x, y)$ stands for $T(xy + yx) - T(y)\theta(x) - \theta(x)T(y)$. Our next task is to prove the following relation

$$T(xy + yx) = T(y)\theta(x) + \theta(x)T(y), \quad \forall\, x \in R. \tag{28}$$

In order to prove it we need the relations below

$$\theta(x)A(x, y)\theta(x) = 0, \quad \forall\, x \in R, \tag{29}$$

and

$$[A(x, y), \theta(x)] = 0, \quad \forall\, x \in R. \tag{30}$$

Let us first prove relation (29). The substitution $xy + yx$ for y in (2) gives

$$T(x^2yx + xyx^2) = \theta(x)T(xy + yx)\theta(x), \quad \forall\, x, y \in R. \tag{31}$$

On the other hand we obtain, by putting $z = x^2$ in (6),

$$T(x^2yx + xyx^2) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x), \quad \forall\, x, y \in R. \tag{32}$$

By comparing (31) and (32) we arrive at (29).

Substituting x by $x+z$ in relation (29) and using (29) again we get for every $x, y, z \in R$ that

$$\theta(x)A(x, y)\theta(z) + \theta(x)A(z, y)\theta(x) + \theta(x)A(z, y)\theta(z)$$
$$+\theta(z)A(x, y)\theta(x) + \theta(z)A(x, y)\theta(z) + \theta(z)A(z, y)\theta(x) = 0$$

. Putting now $-x$ for x in this expression and comparing the relation so obtained with the above one, we obtain $\theta(x)A(x, y)\theta(z)+\theta(x)A(z, y)\theta(x)+\theta(z)A(x, y)\theta(x) = 0$, for every $x, y, z \in R$. Right multiplication of this relation by $A(x, y)\theta(x)$ gives, because of (29),

$$\theta(x)A(x, y)\theta(z)A(x, y)\theta(x) = 0, \quad \forall\, x, y, z \in R. \tag{33}$$

Now, let us proving relation (30). The linearization of (4) gives

$$[T(x), \theta(y)] + [T(y), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{34}$$

Putting $xy + yx$ for y in the above relation and using (4) we obtain $[T(x), \theta(xy + yx)] + [T(xy + yx), \theta(x)] = \theta(x)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x) + [T(xy + yx), \theta(x)] = 0$, for all $x, y \in R$. Thus we have $[T(xy + yx), \theta(x] + \theta(x)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x) = 0$, for all $x, y \in R$. According to (34) we can replace $[T(x), \theta(y)]$ by $-[T(y), \theta(x)]$ in this expression. Therefore, $[T(xy+yx), \theta(x)]-\theta(x)[T(y), \theta(x)]-[T(y), \theta(x)]\theta(x) = 0$, for all $x, y \in R$, which can be rewritten in the form $[T(xy+yx)-T(y)\theta(x)-\theta(x)T(y), \theta(x)] = 0$, for every $x, y \in R$. The proof of relation (30) is therefrom complete.

Relation (30) makes it possible to replace in (33) $\theta(x)A(x, y)$ by $A(x, y)\theta(x)$. Thus we have

$$A(x, y)\theta(x)\theta(z)A(x, y)\theta(x) = 0, \quad \forall\, x, y, z \in R, \tag{35}$$

whence, by the surjectivity of θ and the semiprimeness of R, it follows that

$$A(x,y)\theta(x) = 0, \quad \forall\, x, y \in R. \tag{36}$$

Of course we also have,

$$\theta(x)A(x,y) = 0, \quad \forall\, x, y \in R. \tag{37}$$

The linearization of (36) with respect to x gives $A(x,y)\theta(z) + A(z,y)\theta(x) = 0$, for all $x, y, z \in R$.

Right multiplication of the above relation by $A(x,y)$ gives, because of (37), $A(x,y)\theta(z)A(x,y) = 0$, for all $x, y, z \in R$, which, by the surjectivity of θ and the semiprimeness of R, gives $A(x,y) = 0$, for every $x, y \in R$. The proof of relation (28) is therefrom complete, too.

In particular for $x = y$ relation (30) reduces to $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$, for all $x \in R$.

Combining the above relation with (4) we arrive at $T(x^2) = T(x)\theta(x)$, for all $x \in R$, and $T(x^2) = \theta(x)T(x)$, for every $x \in R$, since R is 2-torsion-free.

By [1, Theorem 2] it follows that T is a left and also right $\theta-$centralizer, which completes the proof.

In particular, we get [4, Theorem 1] as a corollary.

Corollary 3.2 *Let R be a 2-torsion free semiprime ring and let $T\colon R \to R$ be an additive mapping. Suppose that $T(xyx) = xT(y)x$ holds for all $x, y \in R$. In this case T is a centralizer.*

We conclude by proving Theorem 2.2.

Proof of Theorem 2.2. Let 1 denote the identity element of R. By assumption, relation (3) holds for every $x \in R$. Putting $x + 1$ for x in (3) we obtain, for every $x \in R$,

$$3T(x^2) + 2T(x) = T(x)\theta(x) + \theta(x)T(x) + \theta(x)a\theta(x) + a\theta(x) + \theta(x)a, \tag{38}$$

where a stands for $T(1)$. Replacing x by $-x$ in (38) and comparing the relation so obtained with the above one, we obtain

$$6T(x^2) = 2T(x)\theta(x) + 2\theta(x)T(x) + 2\theta(x)a\theta(x),\ \forall\, x \in R. \tag{39}$$

From (39) and since R is 2-torsion-free we have

$$3T(x^2) = T(x)\theta(x) + \theta(x)T(x) + \theta(x)a\theta(x),\ \forall\, x \in R.$$

Substituting from the above relation in (38) we get

$$2T(x) = a\theta(x) + \theta(x)a,\ \forall\, x \in R. \tag{40}$$

We intend to prove that $a \in Z(R)$. According to (40) one can replace $2T(x)$ on the RHS of (39) by $a\theta(x) + \theta(x)a$ and $6T(x^2)$ on the LHS by $3a\theta(x^2) + 3\theta(x^2)a$, to get

$$a\theta(x^2) + \theta(x^2)a - 2\theta(x)a\theta(x) = 0, \ \forall\, x \in R.$$

The above relation can be rewritten in the form

$$[[a, \theta(x)], \theta(x)] = 0, \ \forall\, x \in R. \tag{41}$$

The linearaization of (41) gives

$$[[a, \theta(x)], \theta(y)] + [[a, \theta(y)], \theta(x)] = 0, \ \forall\, x, y \in R. \tag{42}$$

Putting xy for y in (42) we obtain, because of (41) and (42) that, for every $x, y \in R$,

$$\begin{aligned} 0 &= [[a, \theta(x)], \theta(xy)] + [[a, \theta(xy)], \theta(x)] = \\ &= [[a, \theta(x)], \theta(x)]\theta(y) + \theta(x)[[a, \theta(x)], \theta(y)] + \\ &\quad + [[a, \theta(x)]\theta(y), \theta(x)] + [\theta(x)[a, \theta(y)], \theta(x)] = \\ &= \theta(x)[[a, \theta(x)], \theta(y)] + [[a, \theta(x)], \theta(x)]\theta(y) + \\ &\quad + [a, \theta(x)][\theta(y), \theta(x)] + \theta(x)[[a, \theta(y)], \theta(x)] = \\ &= [a, \theta(x)][\theta(y), \theta(x)]. \end{aligned}$$

Thus we have $[a, \theta(x)][\theta(y), \theta(x)] = 0$, for each $x, y \in R$. The substitution $\theta(y)a$ for $\theta(y)$ on this relation gives $[a, \theta(x)]\theta(y)[a, \theta(x)] = 0$, for all $x, y \in R$. So, by the semiprimeness of R and the surjectivity of θ it follows $a \in Z(R)$, which reduces (40) to the form $T(x) = a\theta(x)$, for every $x \in R$. The proof is now complete.

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