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# ON FREE ACTIONS AND DEPENDENT ELEMENTS IN RINGS

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**Abstract.** This paper contains two directions of work. The first one gives material related to the free action mappings and the dependent elements related to those mappings. The other direction deal with a generalization of the definition of dependent elements and free actions. We concentrate our study on dependent elements, free action maps, and those which satisfy $f(x)a = bx$, $\forall x \in R$ and some fixed $a, b \in R$. In the first part we work with one dependent element. That is there exists a fixed element $a \in R$ such that $f(x)a = ax$ $\forall x \in R$. In the second one, we characterize the two elements $a$ and $b$ which have the property $f(x)a = bx$, $\forall x \in R$, when $f$ is assumed to have additional properties like generalized derivation, left centralizer, or is a composition of some well-known maps.

## 1. Introduction

Let $R$ and $S$ be associative rings. An *additive map* $f$: $R \to S$ is a map that preserves addition [4]. That is, $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. We recall that a ring $R$ is said to be *prime* if whenever $a, b \in R$ and $aRb = (0)$

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then $a = 0$ or $b = 0$ [3]. Let now $S \subseteq R$ a nonzero subset of $R$. Then, following [12], $L(S) = \{r \in R : \ rS = (0)\}$ is called the *left annihilator* of $S$ on $R$, and $R(S) = \{r \in R : \ Sr = (0)\}$ is called the *right annihilator* of $S$ on $R$. If $R$ is prime, the right annihilator of a nonzero right ideal of $R$ must be zero. Following [4], a ring $R$ is said to be *semiprime* if whenever $a \in R$ and $aRa = (0)$, then $a = 0$. A semiprime ring has no nonzero nilpotent ideals. Moreover, if there is in a ring $R$ a least positive integer $n$ such that $na = 0$, for all $a \in R$, then $R$ is said to have *characteristic* $n$. If no such $n$ exists, $R$ is said to have *characteristic zero*. We use the usual abbreviation $char(R)$ for the characteristic of $R$. Following [3], a ring $R$ is said to be $n$*-torsion-free*, where $n \neq 0$ is an integer, if whenever $x \in R$ and $nx = 0$, then $x = 0$.

An element $a \in R$ is said to be a *dependent element* of a mapping $F$: $R \to R$ in case $F(x)a = ax$ holds for all $x \in R$ [6]. Let now $b \in R$ be another element of $R$. Then $a, b \in R$ are said to be *two dependent elements* on the map $F$ if $F(x)a = bx$ for all $x \in R$ [2]. Furthermore, a mapping $F$: $R \to R$ is called a *free action* if zero is the only dependent element of $F$ [11].

J. Vukman [10] studied the dependent elements in prime and semiprime rings. He gave the notation of $n$*-dependent*, *Jordan dependent* and *Lie dependent* elements, proving the following: let $R$ be a non commutative semiprime ring and $D$: $R \to R$ be a derivation. If $D([x, y])a = a[x, y]$ holds for all pairs $x, y \in R$ and some $a \in R$, then $a \in Z(R)$, the *center* of $R$, where $[x, y] = xy - yx$ denotes the Lie product as usual. Recall that a map $d$: $R \to R$ is a *derivation* of a ring $R$ if $d$ is additive and satisfies the Leibnitz' rule; $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$ (see [1] for a partial bibliography).

In [11], Vukman and I. Kosi-Ulbl gave a lot of results concern dependent elements and free action for serval mappings in prime and semiprime rings. They proved that in a semiprime ring $R$, there are no nonzero elements which are dependent on the mapping $\alpha + \beta$, where $\alpha$ and $\beta$ are automorphisms of $R$.

An additive mapping $x \to x^\star$ on a ring $R$ is called an *involution* if $(x^\star)\star = x$ and $(xy)^\star = y^\star x^\star$ holds for all $x, y \in R$. A ring equipped with an involution is called a *ring with involution*, or a $\star$*-ring* [3]. Vukman [9] proved the following result: let $R$ be a semiprime $\star$-ring and $D$: $R \to R$ be a derivation. Then the map $F$ defined by $F(x) = D(x^\star)D(x)^\star$, $x \in R$ is a free action. He also proved that any Jordan $\star$-derivation on a 2-torsion-free semiprime $\star$-ring is a free action. Recall that an additive mapping $D$: $R \to R$, where $R$ is a $\star$-ring, is called a $\star$*-derivation* in case $D(xy) = D(x)y^\star + xD(y)$ holds for all $x, y \in R$. And $D$ is called a *Jordan* $\star$*-derivation* if $D(x^2) = D(x)x^\star + xD(x)$ holds for all $x \in R$ [8].

Recently, in [2], Bashammakh also gave some results related to free action mappings and two dependent elements.

In this work we concentrate our study on dependent elements, free action maps, and those maps which satisfy $f(x)a = bx$, for all $x \in R$ and some fixed $a, b \in R$. In the first section we work with one dependent element. That is there exists a fixed element $a \in R$ such that $f(x)a = ax$ for all $x \in R$. In Section two, we characterize the two elements $a$ and $b$ which have the property $f(x)a = bx$, for all $x \in R$, when $f$ is assumed to have additional properties like generalized derivation, left centralizer, or is a composition of some well know maps.

## 2. On dependent elements

In this section we study the dependent elements of a map $f\colon R \to R$, where $R$ is a ring. The *dependent elements* are the elements $a \in R$ which satisfy the relation $f(x)a = ax$, for all $x \in R$. If there exist no nonzero element $a$ which satisfies that relation, we call the map $f$ a *free action*. Free action maps also will be studied in this section.

The main idea is to study the functions satisfying $f(x)a = ax$ for all $x \in R$ and some fixed element $a \in R$.

We begin with two examples

**Example 1**. If $R$ has an identity element and $a$ is invertible, we can characterize all functions with dependent element $a$. The function $f(x) = axa^{-1}$ has $a$ for dependent element, because $f(x)a = axa^{-1}a = ax$. Furthermore, if $f(x)a = ax$ for all $x \in R$ then $f(x) = axa^{-1}$.

**Example 2**. If $Ra = R$ then given any $x \in R$, there exists some $z \in R$ such that $za = ax$. Let $f(x)$ pick one such $z \in R$. Then $f(x)a = za = ax$. Notice that $f$ is not necessarily additive nor multiplicative.

Now we will give some propositions needed for the main theorems:

**Proposition 2.1.** *Given $f\colon R \to R$ with dependent element $a$, then $I = \{x \in R \mid ax = 0\}$ is an ideal of $R$.*

**Proof.** It follows that $a(Rx + xR) = aRx + axR = f(R)(ax) + (ax)R = 0.\square$

**Remark 1**. If $f$ is not $1-1$, then $f(x) = f(y)$ implies $x - y \in I$, for all $x, y \in R$.

In fact, if $f(x) = f(y)$ then $0 = (f(x) - f(y))a = f(x)a - f(y)a = ax - ay = a(x - y)$. So $x - y \in I$.

**Remark 2**. $I = \{x \mid ax = 0\} = \{x \mid f(x)a = 0\}$.

**Remark 3**. The kernel of $f$ is contained in $I$.

In fact, if $f(x) = 0$, then $0 = f(x)a = ax$ so $x \in I$.

**Proposition 2.2.** *The set $H = \{x \mid a^n x = 0$ for some $n\}$ is an ideal of $R$.*

**Proof**. If $a^i x = 0$ and $a^j y = 0$, then

$$\begin{aligned} a^{i+j}(x + y) &= a^j(a^i x) + a^i(a^j y) = 0, \\ a^i(xr) &= (a^i x)r = 0, \text{ and} \\ a^i(rx) &= f^i(r)a^i x = 0. \end{aligned}$$

That is, $H$ is an ideal of $R$.$\square$

We now look at annihilators of $a$ on the other side. There are a lot of them. But they do not form an ideal.

**Proposition 2.3.** *Given* $f\colon R \to R$ *with dependent element* $a \in R$, *then* $\{f(xy) - f(x)f(y)\}a = 0$ *and* $\{f(x+y) - f(x) - f(y)\}a = 0$.

**Proof**. Direct calculations give

$$\begin{aligned}
\{f(xy) - f(x)f(y)\}a &= f(xy)a - f(x)f(y)a \\
&= axy - f(x)ay \\
&= axy - axy = 0, \text{ and} \\
\{f(x+y) - f(x) - f(y)\}a &= f(x+y)a - f(x)a - f(y)a \\
&= a(x+y) - ax - ay = 0. \ \square
\end{aligned}$$

**Definition 2.4.** We will denote $W = \{w \in R \mid wa = 0\}$.

**Proposition 2.5.** *Let* $W$ *as above. Then* $W$ *is a left ideal of* $R$.

**Proof**. It is easy to see that $(RW)a = R(Wa) = 0$ therefore $RW \subset W$.□

**Proposition 2.6.** $WRa$ *is a two sided ideal of* $R$.

**Proof**. We have $R(WRa) \subset (RW)Ra \subset WRa$, because $W$ is a left ideal.

Similarly, $(WRa)R \subset WRaR \subset WRf(R)a \subset WRa$.□

**Proposition 2.7.** $WRa^n$ *is an ideal of* $R$, *for any* $n \in I\!N$.

**Proof**. We have $R(WRa^n) \subset (RW)Ra^n \subset WRa^n$, since $W$ is a left ideal of $R$. Also, $(WRa^n)R \subset WRa^nR \subset WRf^n(R)a^n \subset WRa^n$.□

**Remark 4**. There is a decreasing chain of two-sided ideals

$$WR \supseteq WRa \supseteq WRa^2 \supseteq WRa^3 \supseteq \ldots \supseteq WRa^n \supseteq WRa^{n+1} \supseteq \ldots$$

**Proposition 2.8.** *Let* $f\colon R \to R$ *be a homomorphism with dependent element* $a \in R$, *then* $f(W)a^2 = 0$, *where* $W$ *is defined in Definition 2.4.*

**Proof**. Directly, $f(W)a^2 \subset f(W)f(a)a \subset f(Wa)a \subset f(0)a = a \cdot 0 = 0$.□

**Proposition 2.9.** *Let* $f\colon R \to R$ *be a homomorphism with dependent element* $a \in R$ *and* $W$ *as defined in Definition 2.4. Then* $f^n(W)a^{n+1} = 0$, *for any* $n \in I\!N$.

**Proof**. We proceed by induction.

Assume that $f^n(W)a^{n+1} = 0$ for a fixed $n \in I\!N$. The result is true for $n = 1$. Therefore,

$$f^{n+1}(W)a^{n+2} \subset f(f^n(W))aa^{n+1} \subset f(f^n(W))f(a^{n+1})a \subset$$
$$f(f^n(W)a^{n+1})a \subset f(0)a = a \cdot 0 = 0$$

and the proof is complete.□

**Remark 5**. If $aR = R$, then $WR = 0$ and $W$ is a trivial ideal of $R$.

**Remark 6**. If $f(R) = R$, then $WR = Wf(R) \subset W$ and $W$ is an ideal of $R$.

**Remark 7**. $f(I) \subset W$.

In fact, $f(I)a \subset aI = 0$.

**Remark 8**. We actually know very little about $f$. If we define $f'(x) = f(x) + w(x)$ for any choice of $w(x) \in W$, then $f'$ also works for the dependent element $a$, that is $f'(x)a = (f(x) + w(x))a = f(x)a = ax$.

**Remark 9**. Let $W$, $f$ be as defined above. Then $Q = \{x \in R \mid Wx \subset W\}$ is a subring of $R$. Moreover, $\{a\}$, $f(R)$ and $W \subseteq Q$.

In fact, if $Wx \subset W$ and $Wy \subset W$ then $W(x + y) \subset Wx + Wy \subset W$, and $W(xy) \subset (Wx)y \subset W$. Therefore $Q$ is a subring of $R$.

Since $Wa = 0$ so $Wa \subset W$.

Also, $Wf(R)a \subset WaR \subset (Wa)R = 0$, since $Wa = 0$. Thus $Wf(R) \subset W$.

Moreover $W^2 \subset W$, since $W$ is a left ideal of $R$.

**Remark 10**. The subring $Q$ contains $a$. The restriction of $f$ maps $Q$ into $Q$. And of course, $f(q)a = aq$, for all $q \in Q$.

In $Q$, the set $W$ is a two sided ideal. It is not obvious that we can define $f$ on the equivalence classes of $Q/W$ because we do not know that $W$ is invariant. Yet is should be possible because modifying $f$ by elements in $W$ always yields a valid function with dependent element $a$.

Pick any representative from each equivalence class. Then:

$$\begin{aligned} f'(x+W)(a+W) &= (f(x+w)+W)(a+W) \\ &= f(x+w)a + W \\ &= (f(x)+f(w))a + W \\ &= f(x)a + W, \end{aligned}$$

because $f(W)a \subset W$.

**Proposition 2.10.** *Let* $\Omega = \{x \in W \mid xR \subset W\}$. *Then* $\Omega$ *is an ideal of* $R$.

**Proof**. Let $x \in \Omega$. We will show that $xRin\Omega$.

By definition of $\Omega$ we have $xR \subset W$. Then $(xR)R \subset x(RR) \subset xR \subset W$.

Let $x \in \Omega$. We will show that $Rx \in \Omega$.

It follows that $Rx \subset W$ because $x \in W$ and $W$ is a left ideal of $R$. Then $(Rx)R \subset R(xR) \subset RW \subset W$.

This shows that $\Omega$ is a two sided ideal of $R$.□

**Proposition 2.11.** *$f(R)a = aR$ is a right ideal of $R$.*

**Proof**. It is clear that $f(R)aR \subset f(R)f(R)a \subset f(RR)a \subset f(R)a$.$\square$

**Proposition 2.12.** *$Ra$ is a two sided ideal of $R$.*

**Proof**. It is clear that $R(Ra) \subset (RR)a \subset Ra$, and then

$$(Ra)R \subset R(aR) \subset R(f(R)a) \subset (Rf(R))a \subset Ra.$$

This completes the proof. $\square$

Suppose that $a$ and $b$ are both dependent elements for $f$.
Thus $f(x)(a+b) = f(x)a + f(x)b = ax + bx = (a+b)x$. So the sum of two dependent elements is also a dependent element. Since $f(f(x))ab = af(x)b = abx$, then the product of two dependent elements is a dependent element for $f(f(x))$. Consequently, $f^n(x)a^n = a^n x$.

This yields the following descending chain of ideals:

$$Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq Ra^4 \supseteq \ldots \supseteq Ra^n \supseteq Ra^{n+1} \supseteq \ldots.$$

**Theorem 2.13.** $(Ra)^n \subset Ra^n$.

**Proof**. $Ra$ is an ideal of $R$. then $RaRa \subset RaRa \subset Rf(R)aa \subset Ra^2$.

Using induction, we have that

$$(Ra)^{n+1} \subset Ra(Ra)^n \subset RaRa^n \subset Rf(R)aa^n \subset Ra^{n+1}$$

and this completes the proof.$\square$

**Theorem 2.14.** *If $a$ is nilpotent, then $Ra$ is a nilpotent ideal. In a semiprime ring, if $a$ is nilpotent, then $a = 0$.*

**Remark 11.** Look at the subring generated by $a$ and the function $f$.

One can write the basis as products of the form $a_k b_1 b_2 b_3 \ldots b_n$, where the $b_i$s are defined inductively.

The original $B_i$s are $B_1 = \{f(a), f(a^2), f(a^3), \ldots, f(a^n), \ldots\}$ and by taking all finite products. Now close $B_1$ by taking all images by $f$. This is $B_2$. Close $B_2$ by taking products and then all images by $f$, etc. When two monomials are multiplied, the results are simplified by bringing all of the singleton $a$s to the left end.

**Remark 12.** Left multiplication by $a$ is $1-1$ but not onto.

**Remark 13.** Right multiplication by $a$s collapses things.

If one multiplies by a large enough power of $a$, the whole term collapses to a single power of $a$.

**Remark 14.** If $R$ is $k$-torsion free, where $k$ is a fixed number, and $f$ satisfies some dependence relation which is homogeneous and such that the coefficients do not add up to zero, then $a$ generates a trivial ideal.

In fact, let $k$ be the sum of the coefficients of the dependence relation. Multiplying the dependence relation by a high enough power of $a$ produces $ka^n = 0$. The ideal $aR = 0$ is nilpotent. In a semiprime ring, $a = 0$.

Assume that $f(x)a = ax$. Since $pf(x)p^{-1}pa = pf(x)a = pax$, we find that $g(x) = pf(x)p^{-1}$ is a function that has dependent element $pa$. If $R$ is a ring of $n \times n$ matrices, by the above statement we may assume that $a$ is in row canonical form since $g(x)pa = pax$ for all $x \in R$.

If $a$ is invertible then $f(x) = axa^{-1}$ and we know $g$ exactly. If $a$ is not invertible, then the row canonical form of $a$ is not the identity matrix.

Suppose that the $i^{th}$ column of $a$ does not contain a stair step one. The product $aE_{ii}$ has the same $i^{th}$ column of $a$ and all other entries are zero.
This matrix $aE_{ii}$ can not be obtained by any left multiplication. Thus all elements $aE_{ii}$ must be zero and the $i^{th}$ column of $a$ all zeros.
But we can put this column of zeros anywhere in the matrix by a right multiplication. We cannot do that by a left multiplication. Therefore $a$ can have no nonzero entries. Therefore $a = 0$. So for the ring of $n \times n$ matrices, any nonzero function is a free action.

Assume $A$ is a prime ring and $f(x)a = ax$. Then $Ra$ is an ideal of $R$ since $RaR \subset Rf(R)a \subset Ra$. If $aw = 0$ then $(RaR)(RwR) \subset RawR = 0$, and so in the prime case, $w = 0$ (*).

If $f(p) = 0$ then $f(p)a = ap = 0$ and so $p = 0$.
If $f(p) = f(q)$ then $0 = (f(p) - f(q))a = a(p - q)$, so $p = q$. Therefore in a prime ring $f$ is $1 - 1$.

If $f(p)a = 0$ then $ap = 0$ and so $p = 0$.
If $W = \{x \in R \mid xa = 0\}$ then $W$ is a left ideal of $R$ and $RWf(R) \subset W$ and $f(R) \cap W = 0$, because $f(x) \in W$ means $f(x)a = 0$ then $0 = f(x)a = ax = 0$. Since $ax = 0$, then from (*) $x = 0$. Since $x = 0$ then $f(x) = 0$. Therefore $f(R) \cap L(a) = 0$, where $L(a)$ is the left annihilator of $a$ in $R$.

$$f(p+q)a = a(p+q) = ap + aq = f(p)a + f(q)a.$$

So $(f(p+q) - f(p) - f(q))a = 0$, so "$f(x)Ra$" is additive.

$$f(pq)a = apq = f(p)aq = f(p)f(q)a.$$

So $(f(pq) - f(p)f(q))a = 0$, so $f(x)$ is sort of multiplicative.

**Theorem 2.15.** *If $R$ is a ring and $aR \subseteq Ra$ for some $a \in R$ then there is a dependent function with dependent element $a$.*

**Proof**. Given any $x \in R$, there exists an element $y \in R$ such that $ax = ya$ since $aR \subseteq Ra$.
Define $g(x) = y$. Then $g$ is the function with dependent element $a$, since $g(x)a = ya = ax$.□

**Corollary 2.16.** *If $aR \subset Sa$ for some subset $S$ of $R$, then $g(R) \subseteq S$.*

**Corollary 2.17.** *Let $f(x)a = ax$ be a dependent function with its dependent element $a \in R$. Then $S = Rangef = \{f(x) \mid x \in R\}$ gives $aR \subset Sa$, which is* *a characterization of all dependent functions with their dependent element.*

The following construction seems to be way too complicated. Take any ring $R$ and any element $a$.

Let $Q =< (x_1, x_2, x_3, x_4, \ldots, x_i, \ldots) >$ where $ax_i = x_{i+1}a$ for all $i$. Then $Q$ is closed under coordinatewise addition and coordinatewise multiplication. If $ay_i = y_{i+1}a$ for all $i \in I\!N$ and $az_i = z_{i+1}a$ for all $\in I\!N$, then $a(y_i + z_i) = (y_{i+1} + z_{i+1})a$ for all $i \in I\!N$. So $a(y_i z_i) = (ay_i)z_i = y_{i+1}az_i = y_{i+1}z_{i+1}a$ for all $i \in I\!N$.

Let $H_i$ be all the coordinates appearing in the $i^{th}$ slot. Then $H_{i+1}a = aH_i$. Also notice that one can start anywhere in the sequence so $H_n \subset H_1$ for all $n \in I\!N$. Then the set of all first coordinates $H = H_1$ is a subring which satisfies $aH \subseteq Ha$, and conversely, if $f(x)a = ax$ then the sequence $(x, f(x), f(f(x)), \ldots)$ has the above property.

$$f^{i+1}(x)a = af^i(x),$$

or

$$H = \{x : \text{ for each } n, \ \exists y_n \text{ such that } a^n x = y_n a^n\}.$$

$H$ is closed under addition and multiplication so is a subring.

Essentially, we are saying that each element in the ring has an infinite number of predessors. They are $\ldots f(f(f(x)), f(f(x)), f(x), x$, i.e. $y_n = f^n(x)$.

**Theorem 2.18.** *If $R$ is a ring with identity $1$ and $a$ is an element in $R$ with left inverse $b$, that is, $ba = 1$, then $g(x) = axb$ is a dependent function with dependent element $a \in R$.*

**Proof**. We have $g(x)a = (axb)a = ax(ba) = ax$.$\square$

**Definition 2.19.** If $R$ is a ring with two dependent functions $f$ and $g$ with the same dependent element $a \in R$, we say that $f$ and $g$ are*equivalent* if $f(x)a = g(x)a$ for all $x \in R$.

It might not be true that $f(x) = g(x)$ for all $x \in R$, but then at least $(f(x) - g(x))a = 0$. If $f$ is a dependent function with dependent element $a$, let $W = \{w \mid wa = 0\}$, then any dependent function $f$ can be modified by adding elements of $W$ to it. It will still be a dependent function with dependent element $a$, i.e., if $u$: $R \to W$, then $g(x) = f(x) + u(x)$ is a dependent function with dependent element $a \in R$,

$$g(x)a = (f(x) + u(x))a = f(x)a = ax.$$

We classify all dependent functions on simple rings with identity element. We show that they are all equivalent to some $g(x) = axb$ for some elements $a$ and $b$. If $a = 0$, then choose $b$ to be zero also. When $a \neq 0$ then $b$ is the left inverse of $a$.

**Theorem 2.20.** *Assume that $R$ is a simple ring with identity element $1$. If there exists a function $f$: $R \to R$ such that $f(x)a = ax$, for all $x \in R$, then $f$*

*is equivalent to* $g(x) = axb$ *for some* $b \in R$.

**Proof**. If $a = 0$ then $f(x)a$ is identically zero and $f(x)$ equivalent to $g(x) = 0x0$. If $a \neq 0$, since $R$ has an identity element 1, $RaR$ is a nonzero ideal of $R$. Since $R$ is simple, $R = RaR$. Since $ras \in Rf(s)a \subset Ra$ we have that $Ra = R$. It follows that there exists an element $b \in R$ such that $ba = 1$. The map $g(x) = axb$ is equivalent to $f(x)$.

$$g(x)a = (axb)a = ax,$$

$$(f(x) - g(x))a = ax - ax = 0.$$

Therefore $f(x)$ and $g(x)$ are equivalent. We have shown that the only dependent functions with nonzero dependent element on a simple ring with identity are derived from elements with left inverses.□

We can define a dependent map for an element $a \in R$ if $a$ has a "partial inverse". That is, if there exists an element $b \in R$ such that $ba$ acts like an identity on $a$, i.e., if $(ba)a = a$.

**Theorem 2.21.** *If* $a, b \in R$ *are such that* $Ra = R$ *and* $ba^2 = a$*, then* $R$ *has a dependent map with dependent element* $a$.

**Proof**. Given $x \in R$, then $x = ya$ for some $y \in R$.

Define $g(x) = ayba$. We claim that $g(x)$ is a dependent function with dependent element $a \in R$. Therefore $g(x)a = (ayba)a = ay(ba^2) = aya = ax$.□

**Theorem 2.22.** *If* $R$ *is a simple ring and* $f$ *is a dependent function with dependent element* $a \in R$ *and* $a \neq 0$ *(we do not assume that* $R$ *has an identity element), then* $Ra = R$ *and there exists an element* $b \in R$ *such* $ba^2 = a$ *and the function* $g$ *defined by* $g(x) = ayba$ *where* $x = ya$ *is equivalent to* $f$

**Proof**. If $RaR \neq 0$, then $RaR = R$. Since $RaR \subset Rf(R)a$ we have $Ra = R$. Then $R = Ra = (Ra)a = Ra^2$, and there is some $b \in R$ such that $ba^2 = a$. The map $g(x) = ayba$ where $ya = x$ is a dependent function with dependent element $a$ since $g(x)a = aybaa = ay(baa) = aya = ax$.

Since $g(x)a = f(x)a = ax$ for all $x \in R$, $g$ and $f$ are equivalent. Therefore we have classified all dependent functions on simple rings in terms of the existence of elements $a$ and $b$ such that $(ba)a = a$.□

## 3. On the relation $f(x)a = bx$

In this section we study some kinds of mappings $f\colon R \to R$ which satisfy the condition $f(x)a = bx$, for all $x \in R$ and some fixed elements $a, b \in R$. This property is a generalization of the definition of one dependent element.

In view of the definition of two dependent elements [2], we prefer to work with the following definition

**Definition 3.1.** Let $R$ be a ring and $f$: $R \to R$ be a mapping. We call that $a$ is *associated* with $b$ in $R$ if $f(x)a = bx$ for all $x \in R$. We shall denote by $A(f) = \{(a, b) \in R \times R : \ f(x)a = bx, \ \forall x \in R\}$ the set of associated pairs of $f$.

**Theorem 3.2.** *Let $A(f)$ as above. Then $A(f)$ is an additive subgroup of $R \times R$.*

**Proof**. Let $(a, b)$, $(c, d) \in A(f)$, $f(x)(a + c) = f(x)a + f(x)c = bx + dx = (b + d)x$. This gives $(a + c, b + d) \in A(f)$. Note that $(0, 0) \in A(f)$. For every $(a, b) \in A(f)$, we have $(-a, -b) \in A(f)$.□

**Theorem 3.3.** *If $(a, b) \in A(f)$ then $(a^n, b^n) \in A(f^n)$ for every $n \in I\!N$.*

**Proof**. We need to prove that $f^n(x)a^n = b^n x$ for every $n \in I\!N$. Using the induction principle.
For $n = 1$ we already have $f(x)a = bx$.
For $n = 2$, $f^2(x)a^2 = \{f(f(x))a\}a = bf(x)a = b^2x$.
Now assuming that $f^k(x)a^k = b^k x$, we get,
$f^{k+1}(x)a^{k+1} = \{f^k(f(x))a^k\}a = b^k f(x)a = b^{k+1}x$.□

For the next theorem, we recall that a *generalized derivation* $g$: $R \to R$ associated with a derivation $d$: $R \to R$ is an additive mapping such that $g(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$ [1].

**Theorem 3.4.** *Let $R$ be a semiprime ring and let $g$: $R \to R$ be a generalized derivation related with a derivation $d$: $R \to R$. If $(a, b) \in A(g)$, then $ab = ba$.*

**Proof**. We have

$$g(x)a = bx, \ \forall x \in R. \tag{1}$$

Replacing $x$ by $xy$ in (1) we get

$$g(x)ya + xd(y)a = bxy, \ \forall x, y \in R. \tag{2}$$

Replacing $y$ by $ay$ and using (2) this gives

$$xd(ay)a + bxya = bxay, \ \forall x, y \in R. \tag{3}$$

This can be rewritten as follows

$$xd(ay)a = bx[a, y], \ \forall x, y \in R. \tag{4}$$

Putting $ax$ instead of $x$ in the above equation gives

$$axd(ay)a = bax[a, y], \ \forall x, y \in R. \tag{5}$$

Left multiplication of (4) by $a$ we get

$$axd(ay)a = abx[a, y], \ \forall x, y \in R. \tag{6}$$

Subtracting equation (5) from (6) we get

$$[a, b]x[a, y] = 0, \ \forall x, y \in R. \tag{7}$$

For $y = b$ equation (7) gives $ab = ba$.□

For the next theorem we need the following well known

**Lemma 3.5.** ([10], Lemma 1.). *Let $R$ be a 2-torsion-free semiprime ring and let $a, b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$, then $axb = bxa = 0$ is fulfilled for all $x \in R$.*

**Theorem 3.6.** *Let $R$ be a 2-torsion-free prime ring and let $d, g$ and $h$ be nonzero derivations of $R$. Suppose $f$ be the mapping $x \to d(g(x)) + h(x)$. If $(a, b) \in A(f)$ then $ba = 0$.*

**Proof**. Let $(a, b) \in A(f)$. Then we have

$$f(x)a = bx, \ \forall x \in R. \tag{8}$$

Using the definition of $f$ we get

$$f(xy) = f(x)y + xf(y) + d(x)g(y) + g(x)d(y), \ \forall x, y \in R. \tag{9}$$

Putting $xa$ for $x$ in (8) and applying (9) we obtain

$$f(x)a^2 + xf(a)a + d(x)g(a)a + g(x)d(a)a = bxa.$$

Applying (8) for the above relation we get

$$xba + d(x)g(a)a + g(x)d(a)a = 0, \ \forall x \in R. \tag{10}$$

Putting $yx$ for $x$ in (10) and applying (10) we obtain

$$d(y)xg(a)a + g(y)xd(a)a = 0, \ \forall x, y \in R. \tag{11}$$

In (11), put $y = a$ and $ax$ for $x$ to obtain

$$d(a)axg(a)a + g(a)axd(a)a = 0, \ \forall x \in R.$$

Applying Lemma 3.5 one can conclude that

$$d(a)axg(a)a = 0, \ \forall x \in R.$$

By the primeness of $R$ we get $d(a)a = 0$ or $g(a)a = 0$. If both of them are zero, relation (10) tells us that $xba = 0$ and this gives $ba = 0$. Now if $d(a)a = 0$ and $g(a)a \neq 0$, relation (11) gives $d = 0$ and again relation (10) gives $ba = 0$. Similarly, if $d(a)a \neq 0$ and $g(a)a = 0$, relation (11) gives $g = 0$ and again relation (10) gives $ba = 0$. So in any case we have $ba = 0$. □

Recall that following B. Zalar [13] an additive mapping $T$: $R \to R$ is called a *left centralizer* in case $T(xy) = T(x)y$ holds for all $x, y \in R$; a *right centralizer* in case $T(xy) = xT(y)$ holds for all $x, y \in R$; and a *centralizer* in case $T$ is both a left and a right centralizer. Note that if $a \in R$ then $La(x) = ax$ is a left centralizer and $Ra(x) = xa$ is a right centralizer.

**Theorem 3.7.** *Let $R$ be a prime ring and let $T$ be a nonzero left centralizer of $R$. If $(a, b) \in A(T)$ then $a \in Z(R)$.*

**Proof**. We have

$$T(x)a = bx, \ \forall x \in R. \tag{12}$$

Replacing $x$ by $xy$ in (12), we get $T(xy)a = bxy$, that is,

$$T(x)ya = bxy, \ \forall x, y \in R. \tag{13}$$

Right multiplication of (13) by $z$, we get

$$T(x)yaz = bxyz, \ \forall x, y, z \in R. \tag{14}$$

Replacing $y$ by $yz$ in (13), we get

$$T(x)yza = bxyz, \ \forall x, y, z \in R. \tag{15}$$

Subtracting (15) from (14), we get

$$T(x)y[a, z] = 0, \ \forall x, y, z \in R. \tag{16}$$

Since $R$ is prime and $T \neq 0$, equation (16) gives $[a, z] = 0$ and this means $a \in Z(R)$.□

**Theorem 3.8.** *Let $R$ be a semiprime ring and let $T$ be a nonzero left centralizer of $R$. If $(a, b) \in A(T)$ then $ab = ba$.*

**Proof**. Replacing $y$ by $a$ in equation (16) we get,

$$bx[a, z] = 0, \ \forall x, z \in R. \tag{17}$$

Replacing $x$ by $ax$ in the above equation we get

$$bax[a, z] = 0, \ \forall x, z \in R. \tag{18}$$

Left multiplication of (17) by $a$ gives,

$$abx[a, z] = 0, \ \forall x, z \in R. \tag{19}$$

Subtracting (18) from (19) and putting $z = b$, we get

$$[a, b]x[a, b] = 0, \ \forall x \in R. \tag{20}$$

Since $R$ is semiprime we get $ab = ba$.□

**Theorem 3.9.** *Let $R$ be a prime ring and let $g$ be a generalized derivation of $R$ related with a derivation $d$ of $R$. Assume that $\theta$: $R \to R$ is such that $\theta(x) = [g(x), x]$. If $(a, b) \in A(\theta)$ then $ba = 0$.*

**Proof**. We have $(a, b) \in A(\theta)$. Then

$$\theta(x)a = [g(x), x]a = bx, \ \forall x \in R. \tag{21}$$

Linearizing the last relation we get

$$[g(x),y]a+[g(y),x]a=0,\ \forall x,y\in R. \tag{22}$$

For $y=x$ the last relation gives

$$2[g(x),x]a=0,\ \forall x\in R. \tag{23}$$

Replacing $y$ by $xy$ in (22) we get, for $\omega=[g(x),xy]a+[g(xy),x]a$,

$$\begin{aligned}\omega &= [g(x),x]ya+x[g(x),y]a+[g(x)y+xd(y),x]a=\\ &= [g(x),x]ya+x[g(x),y]a+[g(x),x]ya+g(x)[y,x]a+x[d(y),x]a=\\ &= 2[g(x),x]ya+x[g(x),y]a+g(x)[y,x]a+x[d(y),x]a=0,\end{aligned}$$

for all $x,y\in R$.
Using (22) in the above relation we get

$$2[g(x),x]ya+x[T(y),x]a+g(x)[y,x]a=0,\ \forall x,y\in R, \tag{24}$$

where $T(x)=d(x)-g(x)$. It is clear that $T$ is a left centralizer on $R$.
Putting $y=a$ in (24) we get

$$x[T(a),x]a+g(x)[a,x]a=0,\ \forall x\in R. \tag{25}$$

Replacing $y$ by $ya$ in (24) and using (24) we get, for $\delta=2[g(x),x]ya^2+x[T(ya),x]a+g(x)[ya,x]a$,

$$\begin{aligned}\delta &= =2[g(x),x]ya^2+x[T(y)a,x]a+g(x)y[a,x]a+g(x)[y,x]a^2=\\ &= 2[g(x),x]ya^2+xT(y)[a,x]a+x[T(y),x]a^2+g(x)y[a,x]a=\\ &= +g(x)[y,x]a^2\{2[g(x),x]ya+x[T(y),x]a+g(x)[y,x]a\}a=\\ &= +g(x)y[a,x]a+xT(y)[a,x]ag(x)y[a,x]a+xT(y)[a,x]a=0.\end{aligned}$$

This gives $g(x)y[a,x]a+xT(y)[a,x]a=0$, for all $x,y\in R$. Putting $y=ay$ we get

$$g(x)ay[a,x]a+xT(a)y[a,x]a=0,\ \forall x,y\in R. \tag{26}$$

Replacing $y$ by $xy$ in (26) gives,

$$g(x)axy[a,x]a+xT(a)xy[a,x]a=0,\ \forall x,y\in R. \tag{27}$$

Left multiplication of (26) by $x$ gives,

$$xg(x)ay[a,x]a+x^2T(a)y[a,x]a=0,\ \forall x,y\in R. \tag{28}$$

Subtracting (28) from (27) gives,

$$[g(x)a,x]y[a,x]a+[xT(a),x]y[a,x]a=0,\ \forall x,y\in R,$$

which can be rewritten as

$$g(x)[a,x]y[a,x]a+[g(x),x]ay[a,x]a+x[T(a),x]y[a,x]a=0,\ \forall x,y\in R. \tag{29}$$

Replacing $y$ by $ay$ in the above equation and using (25) we get

$$g(x)[a,x]ay[a,x]a+[g(x),x]a^2y[a,x]a+x[T(a),x]ay[a,x]a=$$

$$g(x)[a,x]ay[a,x]a + [g(x),x]a^2y[a,x]a - g(x)[a,x]ay[a,x]a =$$

$$[g(x),x]a^2y[a,x]a = bxay[a,x]a = 0.$$

This gives,

$$bxay[a,x]a = 0,\ \forall x,y \in R. \tag{30}$$

Therefore, $bxa = 0$ which means $a = 0$ or $b = 0$; or $[a,x]a = 0$. Specially, we have $[g(a),a]a = 0$ and this gives $ba = 0$.□

Recall that if $R$ is a ring and $\alpha$ is a nonzero mapping of $R$, then an additive mapping $D$: $R \to R$ is called an *$\alpha$-derivation* if $D(xy) = D(x)\alpha(y) + xD(y)$, holds for all $x, y \in R$ [5].

**Theorem 3.10.** *Let $R$ be a prime ring and let $d$ be a nonzero $\alpha-$derivation, where $\alpha$ is a homomorphism of $R$. Assume that $\theta$: $R \to R$ is such that $\theta(x) = d(x) + \alpha(x)$. If $(a,b) \in A(\theta)$ then $(a,b) \in A(\alpha)$ or $ab = ba$.*

**Proof**. Assume $(a,b) \in A(\theta)$ then we have $\theta(x)a = bx$, for all $x \in R$. This gives

$$\{d(x) + \alpha(x)\}a = bx,\ \forall x \in R. \tag{31}$$

Replacing $x$ by $xy$ we get

$$\{d(xy) + \alpha(xy)\}a = d(x)ya + \alpha(x)d(y)a + \alpha(x)\alpha(y)a =$$

$$d(x)ya + \alpha(x)\{d(y) + \alpha(y)\}a = d(x)ya + \alpha(x)by = bxy.$$

So we have,

$$d(x)ya + \alpha(x)by - bxy = 0,\ \forall x,y \in R. \tag{32}$$

Replacing $y$ by $ay$ in (32) and from equation (31) using $d(x)a = bx - \alpha(x)a$ we get,

$$bxya - \alpha(x)aya + \alpha(x)bay - bxay = 0,\ \forall x,y \in R. \tag{33}$$

In (33), replace $y$ by $yb$ to get

$$bxyba - \alpha(x)ayba + \alpha(x)bayb - bxayb = 0,\ \forall x,y \in R. \tag{34}$$

Right multiplication of equation (33) by $b$ we get

$$bxyab - \alpha(x)ayab + \alpha(x)bayb - bxayb = 0,\ \forall x,y \in R. \tag{35}$$

Subtracting equation (34) from equation (35) we get,
$bxy[a,b] - \alpha(x)ay[a,b] = 0$, for all $x, y \in R$, which can be rewritten as

$$\{bx - \alpha(x)a\}y[a,b] = 0,\ \forall x,y \in R. \tag{36}$$

From the above equation and the primeness of $R$ we get $\alpha(x)a = bx$ or $ab = ba$ and the proof is complete.□

**Theorem 3.11.** *Let $R$ be a semiprime ring and let $f$ and $T$ be two left centralizers of $R$. Assume that $\theta$: $R \to R$ is such that $\theta(x) = (f \circ T)(x)$. If*

$(a, b) \in A(\theta)$ *then* $ab = ba$.

**Proof**. Let $(a, b) \in A(\theta)$. Then we have $\theta(x)a = bx$, for all $x \in R$. This gives

$$f(T(x))a = bx, \ \forall x \in R. \tag{37}$$

Putting $x = xy$ in (37) gives,

$$f(T(xy))a = f(T(x)y)a = f(T(x))ya = bxy, \ \forall x, y \in R.$$

Replacing $y$ by $ay$ in the above relation we get $f(T(x))aya = bxya = bxay$, for all $x, y \in R$. This gives

$$bx[y, a] = 0, \ \forall x, y \in R. \tag{38}$$

Left multiplication of (38) by $a$ gives

$$abx[y, a] = 0, \ \forall x, y \in R. \tag{39}$$

Putting $x = ax$ in equation (38) gives

$$bax[y, a] = 0, \ \forall x, y \in R. \tag{40}$$

Subtracting (40) from (39)and taking $y = b$ gives

$$[a, b]x[b, a] = 0, \ \forall x, y \in R. \tag{41}$$

Because of the semiprimeness of $R$ we have $ab = ba$. $\square$

**Corollary 3.12.** *Let* $R$ *be a prime ring and let* $f$ *and* $T$ *be two left centralizers of* $R$*. Assume that* $\theta$*:* $R \to R$ *is such that* $\theta(x) = (f \circ T)(x)$*. If* $(a, b) \in A(\theta)$ *then* $b = 0$ *or* $a \in Z(R)$*.*

**Proof**. Equation (38), in case of $R$ is a prime ring, gives that $b = 0$ or $a \in Z(R)$. $\square$

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