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GENERALIZED HIGHER (U, R)-DERIVATIONS

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Abstract

Let R be a ring not necessarily with an identity element. A well-known result proved by I. N. Herstein concerning derivations in prime rings have been extensively studied by many authors. Also, M. Ferrero and C. Haetinger extended this result to higher derivations. The main purpose of this work is to introduce the concept of (U, R)-derivations and generalized (U, R)-derivations. Then we extend Awtar's theorem to

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higher (U, R)-derivations and to generalized higher (U, R)-derivations by giving corresponding definitions.

1. Introduction

Ring theory is a showpiece of mathematical unification, bringing together several branches of the subject and creating a powerful machine for the study of problems of considerable historical and mathematical importance.

Throughout the discussion, unless otherwise mentioned, R denotes an associative ring having at least two elements. However, R may not have unity. Recall that a ring R is said to be *prime* if the product of any two nonzero ideals of R is nonzero. Equivalently, $aRb = \{0\}$ with $a, b \in R$ implies $a = 0$ or $b = 0$. A ring R is called *semiprime* if it has no nonzero nilpotent ideals. Equivalently, $aRa = \{0\}$ with $a \in R$ implies $a = 0$. For any $x, y \in R$, using its associative product we can induce two new products viz. the *Lie product* $[x, y] = xy - yx$ and the *Jordan product* $x \circ y = xy + yx$. An additive subgroup $U \subset R$ is said to be a *Lie ideal* (resp. *Jordan ideal*) of R if whenever $u \in U$ and $r \in R$, then $[u, r]$ (resp. $(u \circ r)$) is also in U.

Rings with derivations are not the kind of subject that undergoes tremendous revolutions, however they have been studied in many papers in the last 50 years, specially the relationships between derivations and the structure of rings (see [1] for an historical account and examples). Many questions on derivations have been considered during the development of the theory, and they were already generalized in several directions. An additive mapping $d : R \to R$ is called a *derivation* (resp. a *Jordan derivation*) of R if $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$ (resp. $d(a^2) = d(a)a + ad(a)$, for all $a \in R$). Every derivation is obviously a Jordan derivation, but the converse is not true in general.

In 1950's, I. N. Herstein initiated the study of the relationship between the associative and the Jordan and Lie structure of associative rings. We refer the reader to [14], [15], [16], where we can find further references and more detailed explanations concerning the motivation and the background of these researches.

In the year 1957, Herstein proved a classical result in this direction which becomes a jumping point for many workers later. The result to which we refer is namely: if R is a prime ring of characteristic different from 2, then every Jordan derivation of R is a derivation ([13, Theorem 3.1]).

Later on Brešar [6] extended the result to 2-torsion-free semiprime rings. A ring R is said to be 2-*torsion-free* if $2x = 0$, $x \in R$ implies $x = 0$. In a subsequent paper, Brešar gave another proof of this result using Jordan triple derivations. An additive mapping $d : R \to R$ is said to be a *Jordan triple derivation* if $d(aba) = d(a)ba + ad(b)a + abd(a)$, for every $a, b \in R$. He proved that every Jordan triple derivation of a 2-torsion-free semiprime ring is a derivation ([7, Theorem 4.3]). It turns out that every Jordan derivation of a 2-torsion-free ring is a Jordan triple derivation ([17, Lemma 3.5]). This gives another proof of the result of Herstein for 2-torsion-free semiprime rings.

Following Lanski and Montgomery [22], a *Lie ideal* of R is any additive subgroup U of R with $[u, r] \in U$ for all $u \in U$ and $r \in R$. Every ring R has a Lie structure by the bracket product $[x, y]$, for $x, y \in R$. The relationship between usual derivations and Lie ideals of prime rings has been extensively studied in the last 30 years, in particular, when this relationship involves the action of the derivation on Lie ideals . In the main results of this paper we assume that the Lie ideal U verifies $u^2 \in U$, for every $u \in U$. A Lie ideal of this type will be called a *square closed* Lie ideal. Furthermore, if the Lie ideal U is square closed and $U \not\subset Z(R)$, where $Z(R)$ denotes que center of R, then U is called an admissible Lie ideal.

R. Awtar extended the Herstein's theorem to Lie ideals ([4, Theorem]). He proved that if U is a Lie ideal of a prime ring R of characteristic different of 2 such that $u^2 \in U$, for every $u \in U$, and $d : R \to R$ is an additive mapping such that $d|_U$ is a Jordan derivation of U into R, then $d|_U$ is a derivation of U into R.

On the other hand, Shammu in [27, Theorem] extended the Herstein's theorem to Jordan ideals. He proved that if R is a prime ring of characteristic different from 2, U is a Jordan ideal of R and $d : R \to R$ is an additive mapping satisfying the condition $d(ur + ru) = d(u)r + ud(r) + d(r)u + rd(u)$, for all $u \in U$, $r \in R$, then $d(ur) = d(u)r + ud(r)$, for all $u \in U$, $r \in R$.

The notion of generalized derivations on a ring A which was introduced by Brešar [8] is related to a derivation of A. In [24], Nakajima defined another type of generalized derivations without using derivations, and gave some categorical properties of that generalized derivations. When A has an identity element, these two notions coincide. The results in [24] were extended to generalized Jordan and Lie derivations in [23].

Following Hvala [18, page 1447], an additive mapping $F : R \to R$ is called a *generalized derivation* if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. We call an additive mapping $F : R \to R$ a *Jordan generalized derivation* if there exists a derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$ holds for all $x \in R$ ([3, page 7]).

Recently, Ashraf and Rehman [3, Theorem] considered the question of Herstein for a Jordan generalized derivation. They showed that in a 2-torsion-free ring R which has a commutator nonzero divisor, every Jordan generalized derivation on R is a generalized derivation.

On the other hand, in his paper [26], Ribenboim gave some properties of higher derivations of modules. His higher derivation f from an A-module M to M is defined by using a higher derivation $D = (d_t) : A \to A$ and the case of length 1 is nothing but a generalized derivation in the sense of Brešar whenever d_0 is the identity map on A.

Assume that R is an algebra over the rational field $\mathbb{Q}$ and $d : R \to R$ is a derivation. Then, if we put $d_n(x) = \frac{d^n(x)}{n!}$, for every $n \in \mathbb{N}$, we have that

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b), \quad \text{for all } a, b \in R \text{ and } n \geq 1. \tag{1}$$

So d defines a sequence $d_0, d_1, ..., d_n, ...$ such that $d_0 = id_R$, d_1 is a derivation and equation (1) holds. A sequence of additive mappings $D = \{d_0, d_1, ..., d_n, ...\}$ is said to be a *higher derivation* of R (HD, for short) if the above relation (1) holds ([19, Exerc. 4, p. 540]). More precisely, higher derivation in a ring R is a sequence of additive mappings $D = (d_i)_{i \in \mathbb{N}}$ of R satisfying the conditions $d_0 = id_R$ and $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in R$ and for all $n \in \mathbb{N}$. For detailed study and examples we refer to readers [2].

Ferrero and Haetinger extended Herstein's theorem to higher derivations, by using Jordan triple higher derivations ([11, Corollary 1.4]). Also, Haetinger in [12, Theorem 2.1] extended Awtar's result to higher derivations.

In 2000, Nakajima defined a generalized higher derivation [23, Definition 2.3] without using a HD at the viewpoint of [24] and gave some categorical properties which are related in [24]. He also treated generalized higher Jordan and Lie derivations. Later, Cortes and Haetinger [9] extended Ashraf's theorem to generalized higher derivations. They proved that if R is a 2-torsion-free ring which has a commutator right nonzero divisor, then every Jordan generalized higher derivation on R is a generalized higher derivation. Recall that if $F = (f_i)_{i \in \mathbb{N}}$ is a family of additive mappings of R such that $f_0 = id_R$, then F is said to be a *generalized higher derivation* (GHD, for short) if there exists a higher derivation $D = (d_i)_{i \in \mathbb{N}}$ of R such that for every $n \in \mathbb{N}$, we have $f_n(ab) = \sum_{i+j=n} f_i(a) d_j(b)$, for all $a, b \in R$ ([25, Definition 2.3(ii)]). Similarly, if U is a Lie ideal of R, then a family of additive mappings of R, $D = (d_i)_{i \in \mathbb{N}}$, is said to be an *HD of U into R* and a family of additive mappings of R, $F = (f_i)_{i \in \mathbb{N}}$, is said to be a *GHD of U into R* in case that the above corresponding conditions are satisfied for all $a, b \in U$.

In a recent paper, Jung [21] improve Jing and Lu's result ([20, Theorem 3.5]) for generalized Jordan triple derivations to generalized Jordan triple higher derivations, proving that if R is a 2-torsion-free prime ring, then every generalized Jordan triple higher derivation on R is a generalized higher derivation. Further, Faraj [10] obtained a more general result. Let R be a 2-torsion-free prime ring and U be an admissible Lie ideal of R. Then every generalized Jordan triple higher derivation of U into R is a generalized higher derivation of U into R.

In this work, we extend and generalized some results above.

We introduce and study the concept of generalized higher (U, R)-derivation and we extend Awtar's theorem [4, Theorem] by proving that if R is a prime ring, $\text{char}(R) \neq 2$, U is an admissible Lie ideal of R and $F = (f_i)_{i \in \mathbb{N}}$, is a generalized higher (U, R)-derivation of R, then $f_n(ur) = \sum_{i+j=n} f_i(u) d_j(r)$, for all $u \in U$, $r \in R$, $n \in \mathbb{N}$.

2. (U, R)-derivations

Throughout this section, we introduce and study the concept of (U, R)-derivation by proving that if R is a prime ring, $\operatorname{char}(R) \neq 2$, U is a square closed Lie ideal of R and d is a (U, R)-derivation of R, then $d(ur) = d(u)r + ud(r)$, for all $u \in U$, $r \in R$, which extends Awtar's result ([4, Theorem]).

Definition 2.1. Let U be a Lie ideal of a ring R. An additive mapping $d : R \to R$ is said to be a (U, R)-*derivation* ((U, R)-D, for short) of R if $d(ur + su) = d(u)r + ud(r) + d(s)u + sd(u)$, for all $u \in U$, $\forall r, s \in R$.

Example. Let R be a ring of all 2×2 matrices over a commutative ring S. Let $U = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in S\right\}$. Then U is a Lie ideal of R.

Define $d : R \to R$ by $d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. Then d is a (U, R)-D.

Lemma 2.2. *Let R be a 2-torsion-free ring and d be a (U, R)-D of R. Then $d(uru) = d(u)ru + ud(r)u + urd(u)$, for all $u \in U$, $r \in R$.*

Proof. Replace r and s by $(2u)r + r(2u)$ in Definition 2.1. Let

$$w = u((2u)r + r(2u)) + ((2u)r + r(2u))u.$$

Then

$$\begin{aligned} d(w) &= 2(d(u)(ur + ru) + ud(ur + ru) + d(ur + ru)u + (ur + ru)d(u)) \\ &= 2(d(u)ur + d(u)ru + ud(u)r + u^2 d(r) + ud(r)u + urd(u) \\ &\quad + d(u)ru + ud(r)u + d(r)u^2 + rd(u)u + urd(u) + rud(u)). \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} d(w) &= d((2u^2)r + r(2u^2)) + 4d(uru) \\ &= 2(d(u)ur + ud(u)r + u^2 d(r) + d(r)u^2 + rd(u)u + rud(u)) + 4d(uru). \end{aligned} \tag{3}$$

By comparing (2) and (3) and since R is 2-torsion-free, we obtain the desired result. □

If we linearize $d(uru) = d(u)ru + ud(r)u + urd(u)$ on u, then we have the following

Corollary 2.3. *Let R be a* 2*-torsion-free ring and d be a* (U, R)*-*D *of R. Then* $d(urv + vru) = d(u)rv + ud(r)v + urd(v) + d(v)ru + vd(r)u + vrd(u)$, *for all* $u, v \in U$.

Remark. Let d be a (U, R)-D of R. We put $\Phi(u, r) = d(ur) - d(u)r - ud(r)$, for all $u \in U$, $r \in R$.

Lemma 2.4. *Let R be a prime ring*, $char(R) \neq 2$, *U be a square closed Lie ideal of R and d be a* (U, R)*-*D *of R. Then* $\Phi(u^2, r) = 0$, *for all* $u \in U$, $r \in R$.

Proof. By [4, Theorem], we have $\Phi(u, v) = 0$, for all $u, v \in U$. Then

$$0 = \Phi(u, ur - ru) = d(u^2r - uru) - d(u)(ur - ru) - ud(ur - ru)$$

$$= d(u^2r) - d(uru) - d(u)ur + d(u)ru - u(d(u)r + ud(r) - d(r)u - rd(u))$$

$$= d(u^2r) - d(u)ur - ud(u)r - u^2d(r) = \Phi(u^2, r). \quad \square$$

Lemma 2.5. *Let R be a prime ring*, $char(R) \neq 2$, *and U be a nonzero admissible Lie ideal of R. Then U contains a nonzero ideal of R.*

Proof. Since U is a noncentral Lie ideal of R, for some $x, y \in U$, $xy - yx \neq 0$. For any $r \in R$,

$$x(yr) - (yr)x = x(yr) - yxr + yxr - (yr)x = (xy - yx)r + y(xr - rx) \in U.$$

Since U is a square closed Lie ideal of $R, 2y(xr - rx) \in U$ and this leads us $2(xy - yx)r \in U$, for all $r \in R$.

For any $r, s \in R$, we have $(2(xy - yx)r)s - s(2(xy - yx)r) \in U$ and $2(xy - yx)rs \in U$. This implies that $s(2(xy - yx))r \in U$, for all $r, s \in R$.

Let $I = R(2(xy - yx))R$. Then it is clear that I is an ideal contained in U. It remains to show that I is nonzero.

Suppose that $I = 0$. Then since R is a 2-torsion-free and prime ring, $xy = yx = 0$ and this is a contradiction. Therefore, I is a nonzero ideal of R. □

The proof of the next lemma can be found in [11, Lemma 2.4]. But we put it here for the sake of completeness.

Lemma 2.6 [11, Lemma 2.4]**.** *Let R be a prime ring, $char(R) \neq 2$, and $U \not\subset Z(R)$ be a Lie ideal. Then there exist elements $a, b, c \in U$ such that $[a, b, c] = abc - cba \neq 0$.*

Proof. Assume that $[x, y, z] = 0$, for every $x, y, z \in U$. By [22, Lemma 1], there exists $u \in U$ with $u^2 \neq 0$. Also $[u^2, v] = [u, u, v] = 0$, for every $v \in U$, and so by [22, Lemma 8], we have that $u^2 \in Z(R)$. Thus the relation $[x, u^2, z] = 0$ gives $[U, U] = 0$, which contradicts [22, Lemma 7]. □

Lemma 2.7. *Let R be a prime ring, $char(R) \neq 2$, and U be an admissible Lie ideal of R. Then for any $t \in R$, if $tv^2 + v^2t = 0$, for all $v \in U$, $t = 0$.*

Proof. Linearize $tv^2 + v^2t = 0$ on v, then

$$t(uv + vu) + (uv + vu)t = 0, \quad \forall u, v \in U. \tag{4}$$

Replace v by $u + v^2$ in (4), then

$$t(uv^2 + v^2u) + (uv^2 + v^2u)t = 0. \tag{5}$$

Since $tv^2 = -v^2t$ for all $v \in U$, then (5) becomes $0 = tuv^2 - v^2tu - utv^2 + v^2tu = 2(tuv^2 + v^2tu)$. Since R is 2-torsion-free, $tuv^2 + v^2tu = 0$ which give us, $v^2[t, u] = 0$, for all $u, v \in U$. Then $v^2[U, t] = 0$ for all $v \in U$. By Lemma 2.5, U contains a nonzero ideal I of R and this give us $v^2I[R, t] = 0$. Since R is prime, either $v^2I = 0$ or $[R, t] = 0$.

If $v^2I = 0$, then since $I \neq 0$ and by using Lemma 2.6 we get $U = 0$ and this is a contradiction. Therefore, $[R, t] = 0$. Since $tv^2 + v^2t = 0$ and R is 2-torsion-free, $tv^2 = 0$, for all $v \in U$.

If we linearize $tv^2 = 0$ on v, then

$$0 = t(u + v)^2 = t(u^2 + uv + vu + v^2) = t(uv + vu)$$

$$= t(uv + vu)u = tuvu = tuvut = (tu)v(tu), \quad \forall u, v \in U, \forall t \in R.$$

Now by Lemma 2.5, U contains a nonzero ideal I of R. Since R is prime, $tu = 0$. Again by Lemma 2.5 and since R is prime, we get $t = 0$. □

We are in position to extend ([4, Theorem]) to (U, R)-derivations.

Theorem 2.8. *Let R be a prime ring, $char(R) \neq 2$, U be a square closed Lie ideal of R and d be a (U, R)-D of R. Then $d(ur) = d(u)r + ud(r)$, for all $u \in U$, $r \in R$.*

Proof. Since d is a (U, R)-D of R,

$$d(uur + uru) = d(u)ur + ud(ur) + d(ur)u + urd(u). \tag{6}$$

On the other hand,

$$d(uur + uru) = d(u^2r) + d(uru) = d(u^2r) + d(u)ru + ud(r)u + urd(u). \tag{7}$$

Now by using Lemma 2.4, equation (7) becomes

$$d(u^2r + uru) = d(u^2)r + u^2d(r) + d(u)ru + ud(r)u + urd(u)$$

$$= d(u)ur + ud(u)r + u^2d(r) + d(u)ru + ud(r)u + urd(u). \tag{8}$$

By comparing (6) and (8), we get

$$u\Phi(u, r) + \Phi(u, r)u = 0, \quad \forall u \in U, \quad r \in R. \tag{9}$$

By linearizing (9) on u, we have

$$u\Phi(v, r) + v\Phi(u, r) + \Phi(u, r)v + \Phi(v, r)u = 0. \tag{10}$$

Replace v by v^2 in (10), then by using Lemma 2.4, we get $v^2\Phi(u, r) + \Phi(u, r)v^2 = 0$, for all $u, v \in U$, $r \in R$.

Now, if $U \not\subset Z(R)$, then by using Lemma 2.7, we get $\Phi(u, r) = 0$, for all $u \in U$, $r \in R$.

If $U \subset Z(R)$, then since R is 2-torsion-free, $v^2\Phi(u, r) = 0$, for all $u, v \in U$, $r \in R$ and this implies that $0 = cv^2\Phi(u, r) = v^2c\Phi(u, r) = 0$, for all $u, v \in U$, $c, r \in R$. Since R is prime, either $v^2 = 0$ or $\Phi(u, r) = 0$. Since $u \neq 0$, $\Phi(u, r) = 0$, for all $u \in U$, $r \in R$. □

Corollary 2.9. *Let R be a prime ring, $char(R) \neq 2$, and U be a square closed Lie ideal of R. Then every Jordan derivation d of U into R is a derivations of U into R.*

3. Generalized (U, R)-derivations

In this section, we introduce the concept of generalized (U, R)-derivation.

Definition 3.1. Let U be a Lie ideal of a ring R and f be an additive mapping of R into itself. We say that f is a generalized (U, R)-derivation ($G(U, R)$-D, for short) of R if there exists a (U, R)-D d of R such that $f(ur + su) = f(u)r + ud(r) + f(s)u + sd(u)$, $\forall u \in U$, $\forall r, s \in R$.

Example. Let $R = \mathcal{M}_2(S)$ be the ring of all the 2×2 matrices over a commutative ring S. Let $U = \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in S\right\}$. Then U is a Lie ideal of R. Define $f : R \to R$ by $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$, for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. Then there exists a (U, R)-D, d of R which is defined by $d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. Hence f is a $G(U, R)$-D of R.

If there is no possibility of misunderstanding, d will always denote a (U, R)-D of R.

Lemma 3.2. *Let R be a 2-torsion-free ring, U be a Lie ideal of R and f be a $G(U, R)$-D of R. Then for all $u \in U$, $r \in R$, the following statements hold*:

(i) $f(uru) = f(u)ru + ud(r)u + urd(u)$;

(ii) $f(urv + vru) = f(u)rv + ud(r)v + urd(v) + f(v)ru + vd(r)u + vrd(u)$.

Proof. (i) Replace r and s by $(2u)r + r(2u)$ in Definition 3.1, then denoting $w = u((2u)r + r(2u)) + ((2u)r + r(2u))u,$ we have

$$\begin{aligned} f(w) = &\ f(u((2u)r + r(2u)) + ((2u)r + r(2u))u) \\ &= 2(f(u)(ur + ru) + ud(ur + ru) + f(ur + ru)u + (ur + ru)d(u)) \\ &= 2(f(u)(ur + ru) + u(d(u)r + ud(r) + d(r)u + rd(u))) \\ &\quad + 2((f(u)r + ud(r) + f(r)u + rf(u))u + (ur + ru)d(u)). \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} f(w) = &\ f(u((2u)r + r(2u)) + ((2u)r + r(2u))u) \\ &= f((2u^2)r + r(2u^2)) + 4f(uru) \\ &= f(2u^2)r + (2u^2)d(r) + f(r)(2u^2) + rd(2u^2) + 4f(uru) \\ &= 2(f(u)ur + ud(u)r + u^2d(r) + f(r)u^2 + rud(u) + rd(u)u) + 4f(uru). \end{aligned} \tag{12}$$

By comparing (11) and (12) and since R is 2-torsion-free, we are ready.

(ii) A linearization of (11) with respect to u gives us the desired result. □

The following well-known result will be used in the paper ([5, Lemma 4]).

Lemma 3.3 [5, Lemma 4]**.** *Let R be a prime ring, $char(R) \neq 2,$ and U be a Lie ideal of R, $U \not\subset Z(R)$. If $a, b \in R$ and $aUb = 0,$ then either $a = 0$ or $b = 0.$*

Remark. Let f be $G(U, R)$-D of R. For every $u \in U, \ r \in R$ we denote by $\delta(u, r)$ (resp. $\Phi(u, r)$) the element of R defined by $\delta(u, r) = f(ur) - f(u)r - ud(r)$ (resp. $\Phi(u, r) = d(ur) - d(u)r - ud(r)$).

Lemma 3.4. *Let R be a prime ring, $char(R) \neq 2,$ U be an admissible Lie ideal of R and f be a $G(U, R)$-D of R. Then $\delta(u, v) = 0,$ for all $u, v \in U$.*

Proof. Let $x = 4(uvwvu + vuwuv)$. Then by Lemma 3.2(ii),

$$\begin{aligned} f(x) = &\ f((2uv)w(2vu) + (2vu)w(2uv)) \\ &= f(2uv)w(2vu) + (2uv)d(w)(2vu) + 2uvwd(2vu) \\ &\quad + f(2vu)w(2uv) + (2vu)d(w)(2uv) + 2vuwd(2uv). \end{aligned}$$

On the other hand,

$$f(x) = f(u(4vwv)u + v(4uwu)v)$$
$$= f(u)(4vwvu) + ud(4vwv)u + 4uvwvd(u)$$
$$+ f(v)(4uwuv) + vd(4uwu)v + 4vuwud(v).$$

Compare the right hand sides of $f(x)$ and since d is a (U, R)-D of R, then

$$0 = 4(\delta(u, v)wvu + \delta(v, u)wuv + uvw\Phi(v, u) + vuw\Phi(u, v))$$
$$= 4(\delta(u, v)w[u, v] + [u, v]w\Phi(u, v)).$$

Since R is 2-torsion-free and by Theorem 2.8, we have $\delta(u, v)w[u, v] = 0.$ Since U is a noncentral Lie ideal, then by Lemma 3.3, $\delta(u, v) = 0,$ for all $u, v \in U.$ □

Remark. If we replace U by a square closed central Lie ideal in Lemma 3.4, then the lemma is also true.

All is prepared for proving the next

Theorem 3.5. *Let R be a prime ring, $char(R) \neq 2$, U be a square closed Lie ideal of R and f be a $G(U, R)$-D of R. Then $f(ur) = f(u)r + ud(r)$, for all $u \in U$, $r \in R$.*

Proof. From Lemma 3.4 and the last remark,

$$\delta(u, v) = 0, \quad \forall u, v \in U. \tag{13}$$

Replace v by $ur - ru$ in (13), then

$$0 = \delta(u, ur - ru) = f(u^2r) - f(uru) - f(u)(ur - ru) - ud(ur - ru).$$

Since d is a (U, R)-D of R, by Lemma 3.2(ii), we get

$$f(u^2r) = f(u)ur + ud(u)r + u^2d(r). \tag{14}$$

Now, let $x = uur + uru.$ Then by Definition 3.1 and by Theorem 2.8,

$$f(x) = f(u)ur + ud(ur) + f(ur)u + urd(u)$$
$$= f(u)ur + ud(u)r + u^2d(r) + f(ur)u + urd(u). \tag{15}$$

On the other hand, using equation (14) and Lemma 3.2(i), we have

$$f(x) = f(u^2r) + f(uru)$$

$$= f(u)ur + ud(u)r + u^2d(r) + f(u)ru + ud(r)u + urd(u). \quad (16)$$

Compare (15) and (16) to get

$$\delta(u, r)u = 0, \quad \forall u \in U, r \in R. \quad (17)$$

Linearizing (17) on u and using equation (17) itself, we get

$$\delta(u, r)v + \delta(v, r)u = 0. \quad (18)$$

Replace v by v^2 in equation (18). Since $\delta(u^2, r) = 0$, then

$$\delta(u, r)v^2 = 0, \quad \forall u, v \in U, r \in R. \quad (19)$$

If U is a central Lie ideal, then $\delta(u, r) = 0$, for all $u \in U$, $r \in R$. If U is noncentral, then replace v by $u + v$ in equation (19) to get $\delta(u, r)vu = 0$, for all $u, v \in U$, $r \in R$. By Lemma 3.3 and since U is noncentral, then $\delta(u, r) = 0$, for all $u \in U$, $r \in R$. □

If we put $f = d$, we obtain Theorem 2.8 again as a corollary.

4. Higher (U, R)-derivations

The aim of this section is to introduce the concept of higher (U, R)-derivation and to extend our results to this type of maps.

Definition 4.1. Let U be a Lie ideal of a ring R and $D = (d_i)_{i\in\mathbb{N}}$ be a family of additive mappings of R into itself such that $d_0 = id_R$. We say that D is a higher (U, R)-derivation of R $(H(U, R)$-D, for short) if for every $n \in \mathbb{N}$, we have $d_n(ur + su) = \sum_{i+j=n} d_i(u)d_j(r) + d_i(s)d_j(u)$, for all $u \in U$, $r, s \in R$.

Example. Let $R = \mathcal{M}_2(S)$ over a commutative ring S of characteristic 2. Let $U = \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in S\right\}$. Then U is a Lie ideal of R.

For all $n \in \mathbb{N}$, we define $d_n : R \to R$ by

$$d_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \begin{pmatrix} 0 & -b \\ nc & 0 \end{pmatrix}, & \text{if } n = 1, 2 \\ 0, & \text{if } n \geq 3 \end{cases}, \quad \text{for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.$$

Then D is an $H(U, R)$-D.

Lemma 4.2. *Let R be a 2-torsion-free ring and* $D = (d_i)_{i \in \mathbb{N}}$ *be an* $H(U, R)$-D *of R. Then* $d_n(uru) = \sum_{i+j+k=n} d_i(u) d_j(r) d_k(u)$, *for all* $u \in U$, $r \in R$, $n \in \mathbb{N}$.

Proof. Replace r and s by $(2u)r + r(2u)$ in Definition 4.1. Let $w = u((2u)r + r(2u)) + ((2u)r + r(2u))u$. *Then*

$$\begin{aligned} d_n(w) &= 2 \sum_{i+j=n} (d_i(u) d_j(ur + ru) + d_i(ur + ru) d_j(u)) \\ &= 2 \sum_{i+j=n} d_i(u) \sum_{l+t=j} (d_l(u) d_t(r) + d_l(r) d_t(u)) \\ &\quad + 2 \sum_{p+q=i} (d_p(u) d_q(r) + d_p(r) d_q(u) d_j(u)) \\ &= 2 \sum_{i+l+t=n} (d_i(u) d_l(u) d_t(r) + d_i(u) d_l(r) d_t(u)) \\ &\quad + 2 \sum_{p+q+j=n} (d_p(u) d_q(r) d_j(u) + d_p(r) d_q(u) d_j(u)). \end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned} d_n(w) &= d_n((2u^2)r + r(2u^2)) + 4d_n(uru) \\ &= \sum_{i+j=n} d_i(2u^2) d_j(r) + d_i(r) d_j(2u^2) + ud_n(uru) \\ &= 2\left(\sum_{i+j=n} \sum_{r+s=i} d_r(u) d_s(u) d_j(r) + d_i(r) \sum_{e+k=j} d_e(u) d_k(u)\right) + 4d_n(uru) \\ &= 2 \sum_{r+s+j=n} d_r(u) d_s(u) d_j(r) + 2 \sum_{i+e+k=n} d_i(r) d_e(u) d_k(u) + 4d_n(uru). \end{aligned} \tag{21}$$

By comparing (20) and (21) and since R is 2-torsion-free, we get the desired result. □

A linearization of $d_n(uru) = \sum_{i+j+k=n} d_i(u)d_j(r)d_k(u)$ with respect to u gives us the following

Corollary 4.3. *Let R be a 2-torsion-free ring and $D = (d_i)_{i\in\mathbb{N}}$ be an $H(U, R)$-D of R. Then $d_n(urv + vru) = \sum_{i+j+k=n} d_i(u)d_j(r)d_k(v) + d_i(v)d_j(r)d_k(u)$, for all $u \in U,\ r \in R,\ n \in \mathbb{N}$.*

Remark. Let $D = (d_i)_{i\in\mathbb{N}}$ be an $H(U,\ R)$-D of R. For every fixed $n \in \mathbb{N}$, for every $u \in U,\ r \in R$ we denote by the element of R defined by $\Phi_n(u,\ r) = d_n(ur) - \sum_{i+j=n} d_i(u)d_j(r)$.

Lemma 4.4. *Let R be a prime ring, $char(R) \neq 2$, U be an admissible Lie ideal of R and $D = (d_i)_{i\in\mathbb{N}}$ be an $H(U,\ R)$-D of R. Then $\Phi_n(u^2,\ r) = 0$, for all $u \in U$, $r \in R,\ n \in \mathbb{N}$.*

Proof. By [11, Corollary 1.4], we have $\Phi_n(u,\ v) = 0$, for all $u,\ v \in U,\ n \in \mathbb{N}$. Replace v by $ur - ru$ in $\Phi_n(u,\ v) = 0$. Then

$$0 = \Phi_n(u,\ ur - ru) = d_n(u^2r) - d_n(uru) - \sum_{i+j=n} d_i(u)d_j(ur - ru)$$

$$= d_n(u^2r) - d_n(uru) - \sum_{i+j=n}\left(d_i(u)\sum_{l+t=j} d_l(u)d_t(r) - d_l(r)d_t(u)\right)$$

$$= d_n(u^2r) - d_n(uru) - \sum_{i+l+t=n} d_i(u)d_l(u)d_t(r) + \sum_{i+l+t=n} d_i(u)d_l(r)d_t(u)$$

$$= d_n(u^2r) - \sum_{p+t=n}\left(\sum_{i+l=p} d_i(u)d_l(u)\right)d_t(r)$$

$$= \Phi_n(u^2,\ r).$$ □

Theorem 4.5. *Let R be a prime ring, $char(R) \neq 2$, U be an admissible Lie ideal of R and $D = (d_i)_{i\in\mathbb{N}}$ be an $H(U, R)$-D of R. Then $d_n(ur) = \sum_{i+j=n} d_i(u)d_j(r)$, for all $u \in U$, $r \in R$, $n \in \mathbb{N}$.*

Proof. We prove the theorem by induction on $n \in \mathbb{N}$. For any $u \in U$, $r \in R$, $\Phi_0(u, r) = 0$. Also by Theorem 2.8, $\Phi_1(u, r) = 0$. Then we can assume that $\Phi_m(u, r) = 0$ for all $u \in U$, $r \in R$, $m < n$, where $m, n \in \mathbb{N}$. Since D is a $H(U, R)$-D of R,

$$\begin{aligned} d_n(uur + uru) &= \sum_{i+j=n} d_i(u)d_j(ur) + d_i(ur)d_j(u) \\ &= ud_n(ur) + d_n(u)ur + \sum_{i+j=n}^{i,\, j<n} d_i(u)d_j(ur) \\ &\quad + urd_n(u) + d_n(ur)u + \sum_{i+j=n}^{i,\, j<n} d_i(ur)d_j(u) \\ &= ud_n(ur) + d_n(u)ur + \sum_{i+j=n}^{i,\, j<n} d_i(u)\sum_{l+t=n} d_l(u)d_t(r) \\ &\quad + urd_n(ur) + d_n(ur)u + \sum_{i+j=n}^{i,\, j<n}\sum_{p+q=i} d_p(u)d_q(r)d_j(u) \\ &= ud_n(ur) + d_n(u)ur + \sum_{i+l+t=n}^{i,\, l+t<n} d_i(u)d_l(u)d_t(r) \\ &\quad + urd_n(u) + d_n(ur)u + \sum_{p+q+j=n}^{p+q,\, j<n} d_p(u)d_q(r)d_j(u). \end{aligned} \tag{22}$$

On the other hand, $d_n(uur + uru) = d_n(u^2r) + d_n(uru)$. By Lemmas 4.2 and 4.4, we have

$$d_n(uur + uru) = \sum_{i+j=n} d_i(u^2)d_j(r) + \sum_{i+j+k=n} d_i(u)d_j(r)d_k(u)$$

$$= \sum_{i+j=n} \sum_{g+h=i} d_g(u) d_h(u) d_j(r) + \sum_{i+j+k=n} d_i(u) d_j(r) d_k(u)$$

$$= \sum_{g+h+j=n} d_g(u) d_h(u) d_j(r) + \sum_{i+j+k=n} d_i(u) d_j(r) d_k(u)$$

$$= d_n(u)ur + u \sum_{h+j=n} d_h(u) d_j(r) + \sum_{g+h+j=n}^{g, h+j<n} d_j(u) d_h(u) d_j(r)$$

$$+ urd_n(u) + \sum_{i+j=n} d_i(u) d_j(r) u + \sum_{i+j+k=n}^{i+j, k<n} d_i(u) d_j(r) d_k(u). \quad (23)$$

By comparing (22) and (23), we get

$$\Phi_n(u, r)u + u\Phi_n(u, r) = 0, \quad \forall u \in U, r \in R, n \in \mathbb{N}. \quad (24)$$

A linearization of (24) with respect to u gives us

$$\Phi_n(u, r)v + \Phi_n(v, r)u + u\Phi_n(v, r) + v\Phi_n(u, r) = 0, \forall u, v \in U, r \in R, n \in \mathbb{N}. \quad (25)$$

Replace v by v^2 in equation (25). Then using Lemma 2.7, we get $\Phi_n(u, r) = 0$, for all $u \in U$, $r \in R$, $n \in \mathbb{N}$. □

5. Generalized Higher (U, R)-derivations

In this last section, we shall introduce and study the concept of generalized higher (U, R)-derivations and we prove the main theorem of the paper, which extends ([4, Theorem]).

Theorem 5.1. *Let R be a prime ring, $char(R) \neq 2$, U be an admissible Lie ideal of R and $F = (f_i)_{i \in \mathbb{N}}$ be a generalized higher (U, R)-derivation of R. Then $f_n(ur) = \sum_{i+j=n} f_i(u) d_j(r)$, for all $u \in U$, $r \in R$, $n \in \mathbb{N}$.*

We begin with the following

Definition 5.2. Let U be a Lie ideal of a ring R and $F = (f_i)_{i \in \mathbb{N}}$ be a family of additive mappings of R into itself such that $f_0 = id_R$. Then, F is said to be a *generalized higher* (U, R)*-derivation of* R $(GH(U, R)\text{-}D,$ for short$)$ if there exists an $H(U, R)\text{-}D$ of R, $D = (d_i)_{i\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$, we have $f_n(ur + su) = \sum_{i+j=n} f_i(u)d_j(r) + f_i(s)d_j(u)$, for all $u \in U$, $r, s \in R$.

Example. Let $R = \mathcal{M}_2(S)$ over a commutative ring S, $char(S) = 2$.

Let $U = \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in S\right\}$. Then U is a Lie ideal of R.

Let $F = (f_i)_{i\in\mathbb{N}}$ be a family of mappings of R into R defined by

$$f_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \begin{pmatrix} na & 0 \\ 0 & -d \end{pmatrix}, & \text{if } n = 1, 2 \\ 0, & \text{if } n \geq 3 \end{cases}, \quad \text{for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.$$

Then there exists an $H(U, R)\text{-}D$ of R, $D = (d_i)_{i\in\mathbb{N}}$, which was defined earlier. Therefore, F is a generalized higher (U, R)-derivation of R.

Throughout this section $D = (d_i)_{i\in\mathbb{N}}$ will denote an $H(U, R)\text{-}D$ of R.

Lemma 5.3. *Let* R *be a* 2*-torsion-free ring,* U *be a Lie ideal of* R *and* $F = (f_i)_{i\in\mathbb{N}}$ *be a* $GH(U, R)\text{-}D$ *of* R*. Then for every* $u, v \in U$, $r \in R$, $n \in \mathbb{N}$, *the following statements hold*:

(i) $f_n(uru) = \sum_{i+j+k=n} f_i(u)d_j(r)d_k(u)$;

(ii) $f_n(urv + vru) = \sum_{i+j+k=n} f_i(u)d_j(r)d_k(v) + f_i(v)d_j(r)d_k(u)$.

Proof. (i) For a fixed $n \in \mathbb{N}$, let $w = u((2u)r + r(2u)) + ((2u)r + r(2u))$, where $u \in U$, $r \in R$. Since F is a $GH(U, R)\text{-}D$ of R, we have

$$\begin{aligned} f_n(w) &= 2\sum_{i+j=n} f_i(u)d_j(ur + ru) + f_i(ur + ru)d_j(u) \\ &= 2\sum_{i+j=n} f_i(u)\sum_{l+t=j} d_l(u)d_t(r) + d_l(r)d_t(u) \\ &\quad + 2\sum_{i+j=n}\left(\sum_{p+q=i} d_p(u)d_q(r) + d_p(r)d_q(u)\right)d_j(u) \end{aligned}$$

$$= 2\sum_{i+l+t=n} f_i(u)d_l(u)d_t(r) + f_i(u)d_l(r)d_t(u)$$

$$+ 2\sum_{p+q+j=n} d_p(u)d_q(r)d_j(u) + d_p(r)d_q(u)d_j(u). \tag{26}$$

On the other hand,

$$\begin{aligned} f_n(w) &= f_n((2u^2)r + r(2u^2)) + 4f_n(uru) \\ &= \sum_{i+j=n} f_i(2u^2)d_j(r) + f_i(r)d_j(2u^2) + 4f_n(uru) \\ &= 2\sum_{i+j=n}\sum_{g+h=i} f_g(u)d_h(u)d_j(r) \\ &\quad + 2\sum_{i+j=n} f_i(r)\sum_{s+k=j} d_s(u)d_k(u) + 4f_n(uru) \\ &= 2\sum_{g+h+j=n} f_g(u)d_h(u)d_j(r) \\ &\quad + 2\sum_{i+s+k=n} f_i(r)d_s(u)d_k(u) + 4f_n(uru). \end{aligned} \tag{27}$$

By comparing (26) with (27) and since R is 2-torsion-free, we get the desired result.

(ii) A linearization of (i) with respect to u gives us:

$$\begin{aligned} f_n((u+v)r(u+v)) &= 2\sum_{i+j+k=n} f_i(u)d_j(r)d_k(u) + f_i(u)d_j(r)d_k(v) \\ &\quad + 2\sum_{i+j+k=n} f_i(v)d_j(r)d_k(u) + f_i(v)d_j(r)d_k(v). \end{aligned} \tag{28}$$

On the other hand,

$$f_n((u+v)r(u+v)) = f_n(uru) + f_n(urv + vru) + f_n(vrv). \tag{29}$$

By comparing (28) with (29), we get the required result. □

Remark. Let $F = (f_i)_{i\in\mathbb{N}}$ be a $GH(U,\ R)$-D of R. For every fixed $n \in \mathbb{N}$, for every $u \in U$, $r \in R$ we denote by $\delta_n(u, r)$ (resp. $\Phi_n(u, r)$) the element of R defined by

$$\delta_n(u, r) = f_n(ur) - \sum_{i+j=n} f_i(u) d_j(r) \quad (\text{resp. } \Phi_n(u, r) = d_n(ur) - \sum_{i+j=n} d_i(u) d_j(r)).$$

Lemma 5.4. *Let R be a prime ring, $char(R) \neq 2$, U be an admissible Lie ideal of R and $F = (f_i)_{i\in\mathbb{N}}$ be a $GH(U, R)$-D of R. Then $\delta_n(u, v) = 0 = 0$, for all $u, v \in U$, $n \in \mathbb{N}$.*

Proof. For all $u, v \in U$, $\delta_0(u, v) = 0$ and by Lemma 3.4, $\delta_1(u, v) = 0$.

Assume, by induction on $n \in \mathbb{N}$, that $\delta_m(u, v) = 0$ for all $u, v \in U$, $m \in \mathbb{N}$, $m < n$.

Let $x = 4(uvwvu + vuwuv)$, where $w \in U$. Using Lemma 5.3(ii), we have

$$f_n(x) = 4 f_n(uv) wvu + 4uvwd_n(vu) + 4 \sum_{\substack{i+j+k=n \\ i,k<n}} f_i(uv) d_j(w) d_k(vu)$$

$$+ 4 f_n(vu) wuv + 4vuwd_n(uv) + 4 \sum_{\substack{i+j+k=n \\ i,k<n}} f_i(uv) d_j(w) d_k(uv).$$

On the other hand, using Lemma 5.3(i), and since $D = (d_i)_{i\in\mathbb{N}}$ is a $H(U, R)$-D of R, then by Lemma 4.2, we have

$$f_n(x) = 4uvw \sum_{s+k=n} d_s(v) d_k(u) + 4 \sum_{i+p=n} f_i(u) d_p(v) wvu$$

$$+ 4 \sum_{\substack{i+p+q+s+k=n \\ s+k,\, i+p<n}} f_i(u) d_p(v) d_q(w) d_s(v) d_k(u) + 4vuw \sum_{r+k=n} d_r(u) d_k(v)$$

$$+ 4 \sum_{i+l=n} f_i(v) d_l(u) wuv + 4 \sum_{\substack{i+l+t+k=n \\ i+l,\, r+k<n}} f_i(v) d_l(u) d_t(w) d_r(u) d_k(v).$$

Now compare the right hand side of these two expressions of $f_n(x)$. Since R is 2-torsion-free and $\delta_m(u, v) = 0$ for all $u, v \in U$, $m < n$, we get

$$\delta_n(u, v) w[u, v] + [u, v] w \Phi_n(u, v) = 0.$$

By Theorem 4.5, we get $\delta_n(u, v)w[u, v] = 0.$

Now by Lemma 3.3 and since U is noncentral, then $\delta_n(u, v) = 0,$ for all $u, v \in U,$ $n \in \mathbb{N}.$ □

Now we can prove our main theorem

Proof of Theorem 5.1. For all $u \in U,$ $r \in R,$ we have that $\delta_0(u, r) = 0.$ By Theorem 3.5, $\delta_1(u, r) = 0.$

Assume, by induction on $n \in \mathbb{N},$ that $\delta_m(u, r) = 0,$ for all $u \in U,$ $r \in R,$ $m \in \mathbb{N},$ $m < n.$

Since $F = (f_i)_{i\in\mathbb{N}}$ is a $GH(U, R)$-D of $R,$

$$0 = \delta_n(u, ur - ru) = f_n(u^2r) - f_n(uru) - \sum_{i+j=n} f_i(u)d_j(ur - ru).$$

Now, since $D = (d_i)_{i\in\mathbb{N}}$ is an $H(U, R)$-D of R, by Lemma 5.3(i), we get

$$f_n(u^2r) = \sum_{i+l+t=n} f_i(u)d_l(u)d_t(r). \tag{30}$$

Thus, since $F = (f_i)_{i\in\mathbb{N}}$ is a $GH(U, R)$-D of $R,$

$$\begin{aligned} f_n(uur + uru) &= \sum_{i+j=n} f_i(u)d_j(ur) + f_i(ur)d_j(u) \\ &= f_n(u)ur + ud_n(ur) + \sum_{i+j=n}^{i,\, j<n} f_i(u)d_j(ur) \\ &\quad + f_n(ur)u + urd_n(u) + \sum_{i+j=n}^{i,\, j<n} f_i(ur)d_j(u). \end{aligned} \tag{31}$$

Since $\delta_m(u, r) = 0,$ for all $u \in U,$ $r \in R,$ $m < n,$ and using Theorem 4.5, equation (31) reduces to

$$\begin{aligned} f_n(uur + uru) = {} & f_n(u)ur + ud_n(ur) + \sum_{i+l+t=n}^{i,\, l+t<n} f_i(u)d_l(u)d_t(r) \\ & + f_n(ur)u + urd_n(u) + \sum_{p+q+j=n}^{p+q,\, j<n} f_p(u)d_q(r)d_j(u). \end{aligned} \tag{32}$$

On the other hand, by using equation (30) and Lemma 5.3,

$$\begin{aligned} f_n(uur + uru) &= f_n(u^2 r) + f_n(uru) \\ &= \sum_{i+l+t=n} f_i(u)\, d_l(u) d_t(r) + \sum_{i+j+k=n} f_i(u) d_j(r) d_k(u) \\ &= f_n(u)ur + u\sum_{l+t=n} d_l(u) d_t(r) + \sum_{i+l+t=n}^{i,\, l+t<n} f_i(u) d_l(u) d_t(r) \\ &\quad + urd_n(u) + \sum_{i+j=n} f_i(u) d_j(r) u + \sum_{i+j+k=n}^{i+j,\, k<n} f_i(u) d_j(r) d_k(u). \end{aligned} \tag{33}$$

Compare (32) with (33) and use Theorem 4.5 to get

$$\delta_n(u, r)u = 0, \quad \forall u \in U,\ r \in R,\ n \in \mathbb{N}. \tag{34}$$

Linearize equation (34) on u and use it again to get

$$\delta_n(u, r)v + \delta_n(v, r)u = 0, \quad \forall u, v \in U,\ r \in R,\ n \in \mathbb{N}.$$

Replace v by v^2 in the last equation. Since $\delta_n(u^2, r) = 0$, $\delta_n(u, r)v^2 = 0$ and this implies that $0 = \delta_n(u, r)(u + v)^2 = \delta_n(u, r)vu$. Hence by Lemma 3.3 and since $U \neq 0$, $\delta_n(u, r) = 0$ for all $u \in U$, $r \in R$, $n \in \mathbb{N}$. □

If we put $f_i = d_i$ for all $i \in \mathbb{N}$ in the last theorem, then we have the following.

Corollary 5.5. *Let R be a prime ring, $char(R) \neq 2$, U be an admissible Lie ideal of R. Then every generalized Jordan higher derivation of U into R is a generalized higher derivation of U into R.*

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