XVI Coloquio Latinoamericano de Álgebra

On Lie Ideals and Jordan Left Centralizers of 2-Torsion-Free Rings

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The work that I will present here was written jointly with my friend Wagner Cortes during the XVII Brazilian Algebra Meeting (UNICAMP, Brazil, 2004), and take part of a preprint paper.

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Introduction

This research has been motivated by:

1. **Vukman, J.**; *An identity related to centralizers in semiprime rings*, Comment. Math. Univ. Carolinae 40(3) (1999), 447-456.
2. **Vukman, J.; Ulbl, I.K.**; *On centralizers of semiprime rings*, Aeq. Math. 66 (2003), 277-283.
3. **Zalar, B.**; *On centralizers of semiprime rings*, Comment. Math. Univ. Carolinae 32 (1991), 609-614.

Throughout this lecture R will denote an associative ring, with center $Z(R)$, not necessarily with 1, and U denotes a Lie ideal of R.

As usual, $[x, y]$ denotes the commutator $xy - yx$.

Recall that if R is a ring, R has a Lie structure by the bracket product $[x, y]$, for $x, y \in R$.

A *Lie ideal* of R is any additive subgroup U of R with $[u, r] = ur - ru \in U$, $\forall u \in U$, $\forall r \in R$.

An additive mapping $G\colon R \to R$ is called a *left (resp. right) centralizer*, if $G(xy) = G(x)y$ (resp. $G(xy) = xG(y)$) holds $\forall x, y \in R$.

If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer.

We follow B. Zalar ([3]) and call G a *centralizer*, if G is both a left and a right centralizer.

If R is a ring with involution $\star$, then every additive mapping $E\colon R \to R$ which satisfies $E(x^2) = E(x)x^\star + xE(x)$, $\forall x \in R$ is called a *Jordan $\star$-derivation.*

Following B. Zalar ([3]) these mappings are closely connected with a question of representability of quadratic forms by bilinear forms.

M. Brešar and B. Zalar[a] obtained a representation of Jordan $\star$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space.

[a] *On the structure of Jordan $\star$-derivations*, Colloquium Math. 63(2) (1992), 163-171.

If we introduce a new product in an associative ring R, given by $x \circ y = xy + yx$, then:

- *Jordan derivation* is an additive mapping d which satisfies $d(x \circ y) = d(x) \circ y + x \circ d(y)$, $\forall x, y \in R$, and
- *Jordan homomorphism* is an additive mapping h which satisfies $h(x \circ y) = h(x) \circ h(y)$, $\forall x, y \in R$.

Therefore, we can define a Jordan centralizer to be an additive mapping T which satisfies
$T(x \circ y) = T(x) \circ y = x \circ T(y)$, $\forall x, y \in R$.

Since the product $\circ$ is commutative, there is no difference between the left and right Jordan centralizers.

An easy computation shows that every centralizer is also a Jordan centralizer. In ([3]), B. Zalar proves that every Jordan centralizer of a semiprime ring is a centralizer.

So, an additive mapping G: $R \to R$ is called a *left (resp. right) Jordan centralizer*, if $G(x^2) = G(x)x$ (resp. $G(x^2) = xG(x)$) holds $\forall x \in R$.

Similarly, if U is a Lie ideal of R, then an additive mapping G: $R \to R$ is said to be a left (resp. right) Jordan centralizer (centralizer) *of* U *into* R in case that the above corresponding conditions are satisfied for all $x \in U$.

Sometimes in this lecture we will assume that the Lie ideal U verifies $u^2 \in U$, $\forall u \in U$. A Lie ideal of this type will be called a *square closed* Lie ideal.

Let U be a square closed Lie ideal of R. It follows[a] that $uv + vu$ and $2uv \in U$.

We have particular interest in square closed Lie ideals. Such distinction already occurs in some works involving usual derivations[b].

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[a] **Ferrero, M.; Haetinger, C.**;*Higher derivations and a theorem by Herstein*, Quaestiones Math. 25(2) (2002), 249-257.

[b] **Awtar, R.**; *Lie ideals and Jordan derivations of prime rings*, Proc. Amer. Math. Soc. 90(1) (1984), 9-14.

B. Zalar ([3]) proved that any left (resp. right) Jordan centralizer of a 2-torsion-free semiprime ring is a left (resp. right) centralizer.

It is our aim in this preprint work to prove the result above changing the assumption of the semiprimality condition on R by another one.

Furthermore, we state this result for Lie ideals.

Main Results

We prove the following

Theorem 1 *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has a commutator right (resp. left) nonzero divisor, and $G\colon R \to R$ a left (resp. right) Jordan centralizer mapping of U into R. Then G is a left (resp. right) centralizer mapping of U into R.*

Since $U = R$ is obviously a square closed Lie ideal of R, then Theorem 1 is also true for left (resp. right) Jordan centralizer mappings of R.

Let us point out that in case $1 \in R$, Theorem 1 above can be easily proved for $U = R$.

Namely, in this case one puts $x + 1$ for x in $G(x^2) = G(x)x$ (resp. $G(x^2) = xG(x)$), which gives $G(x) = G(1)x$ (resp. $G(x) = xG(1)$).

Thus, G is a left (resp. right) centralizer.

Furthermore, if R is a 2-torsion-free ring, U is a square closed Lie ideal of R, and $G: R \to R$ is a left (resp. right) centralizer of U into R, then an easy computation gives that $G(xyz) = G(x)yz$ (resp. $G(xyz) = xyG(z)$), $\forall x, y, z \in U$.

A natural question is to ask whether the converse is also true:

Theorem 2 *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has a commutator right (resp. left) nonzero divisor, and $G: R \to R$ an additive mapping that satisfies $G(xyz) = G(x)yz$ (resp. $G(xyz) = xyG(z)$), $\forall x, y, z \in U$. Then G is a left (resp. right) centralizer of U into R.*

Finally, if $G: R \to R$ is a left and right centralizing mapping of a ring R, then $2G(xyz) = G(x)yz + xyG(z)$, $\forall x, y, z \in R$. We obtained a result that provides a converse of this fact.

Theorem 3 *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has an element that is nonzero divisor, and $G: R \to R$ an additive mapping that satisfies $2G(xyz) = G(x)yz + xyG(z)$, $\forall x, y, \in U$. Then G is a centralizer of U into R.*

Remarks and Examples

As we wrote above, B. Zalar proved ([1], Proposition 1.4) that if R is a semiprime ring of $char(R) \neq 2$, and $T\colon R \to R$ is and additive mapping which satisfies $T(x^2) = T(x)x$, $\forall x \in R$, then T is a left centralizer.

The following example shows, for the sake of completeness, that the assumptions of our Theorem 1 and Zalar's Proposition are independent each other.

This example is due to M. Brešar who kindly allowed us to include it here.

Example: A semiprime ring may not contain a commutator nonzero divisor (after all, take commutative semiprime rings, or more generally, semiprime rings R containing a nonzero central idempotent element $e \in R$ such that eR is commutative).

Conversely, a ring may contain a commutator nonzero divisor, but is not semiprime. For example, let $R = T_2(A_1)$ be the ring of the 2×2 upper triangular matrices whose entries are elements from the Weyl algebra A_1 (polynomials in x, y such that $xy - yx = 1$). Then R is not semiprime, but the commutator of scalar matrices generated by x and y is the identity matrix.

We give in a recent paper [a] some well known examples of rings that have commutators nonzero divisors.

Example: One can consider any noncommutative ring without zero divisors, the matrix algebra over a division ring.

[a] **Cortes, W.; Haetinger, C.**; *On Jordan generalized higher derivations in rings*, Turkish. J. of Math. 29(1) (2005), 1-10.

Example: Consider the 2×2 matrix algebra $\mathcal{M}_2(D)$ over a domain D with 1. Let E_{ij} be the usual matrix units. Then the commutator $[E_{12}, E_{21}] = E_{11} - E_{22}$ is invertible.

When $char(D) \neq 2$, then in $\mathcal{M}_3(D)$ we have that $[E_{12} + E_{23}, E_{21} - E_{32}] = E_{11} - 2E_{22} + E_{33}$ is a nonzero divisor.

Clearly variations of this will work for $\mathcal{M}_n(D)$ where D is a noncommutative domain (just consider 2×2 block diagonal matrices and a 3×3 block at the bottom if n is odd). This block matrix idea will also work for the ring of all (countably infinite) row and column finite matrices over a domain.

Example: In rings like $R = F\langle x, y\rangle/(x^2)$, the zero divisors must lie in xR or Rx, so $[x, y]$ is regular.

Example: Of course once one has suitable examples of prime rings, then direct sums give examples for semiprime rings.

Remark. If an additive mapping $G\colon R \to R$, where R is an arbitrary ring, is both left and right Jordan centralizer, then obviously G satisfies the relation $2G(x^2) = G(x)x + xG(x)$, $\forall x \in R$.

J. Vukman proved ([1], Theorem 1) that if R is a 2-torsion-free semiprime ring and $G\colon R \to R$ is an additive mapping such that $2G(x^2) = G(x)x + xG(x)$ holds $\forall x \in R$, then G is left and right centralizer.

It seems natural to ask whether the result of ([1]) is also true for Lie ideals. We were unable to answer this question.

Something to Think About

In non-commutative rings, the notion of a derivation is extended to a (σ,τ)-derivation, a right derivation and a central derivation, etc.

There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings.

The properties of (σ,τ)-derivations were discussed in many papers with respect to the ring structures.

For right derivations, M.E. Sweedler (1980) gave some elementary properties of them and using them, he introduced the notion of differentially separability, which is related to separable algebras.

For skew polynomial rings, all left (right) derivations were obtained in a similar way to a polynomial ring (see A. Nakajima and M. Sapanci, 1994).

Later, Hongan and H. Komatsu (1994) constructed the module of differentials of these deformed derivations in a non-commutative algebra with or without identity.

On the other hand, for the generalized derivations which were introduced by M. Brešar (1991), B. Hvala (1998) proved a product property of it on a semiprime ring.

In 1999, A. Nakajima gave an exact sequence of the set of generalized derivations and the set of derivations from a k-algebra S to an S/k-bimodule M. Combining this and the result of the universal mapping property of derivations (cf. N. Bourbaki), he proved the correspondig universal mapping property of generalized derivations. After that, he gave similar results for left derivations and central derivations.

In 2000, A. Nakajima gave some categorical properties which are related to the above, and he treat generalized higher Jordan and Lie derivations.

The challenge is to extend all these results to (left/central) generalized higher derivations.

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