On Left Jordan Centralizers

Claus Haetinger
UNIVATES University Center (Brazil)
XVIII Latin American Algebra Colloquium

São Pedro, August 3^{rd} to August 8^{th}, 2009

Co-Authors

- **M.S. Tammam El-Sayiad** - Beni Suef University (Egypt)
- **M.N. Daif** - Al-Azhar University (Egypt)
- **H.A. El-Saify** - Beni Suef University (Egypt)
- **I.R. Hentzel** - Iowa State University (USA)
- **C. Haetinger** - Univates University Center (Brazil)

This work is based on the Chapter 5 of the Ph.D. thesis of M.S. Tammam El-Sayiad, entitled **Prime and semiprime rings endowed with some kinds of mappings**, Beni suef University (Egypt), 2008.

In this lecture me concentrate our study on Jordan left centralizers.

The first result will characterize rings with a Jordan centralizer T.

Such rings have a T invariant ideal I, and I is the union of an ascending chain of nilpotent ideals.

Also, we include an application of this result.

The material in this chapter 5 is contained in **Hentzel, I.R.; Tammam El-Sayiad, M.S.**: *Left centralizers on rings which are not semiprime*, Rocky Mountain J. Math. (to appear).

Table of Contents

Introduction, Definitions and Preliminary Results

A *(left) centralizer* for an associative ring R is an additive map satisfying

$$T(xy) = T(x)y, \ \forall x, y \in R.$$

A *(left) Jordan centralizer* for an associative ring R is an additive map satisfying

$$T(xy + yx) = T(x)y + T(y)x, \ \forall x, y \in R.$$

We characterize rings with a Jordan centralizer T.

Such rings have a T invariant ideal I.

T is a centralizer on R/I.

And I is the union of an ascending chain of nilpotent ideals.

Our work requires $char(R) \neq 2$.

This result has applications to (right) centralizers, (two sided) centralizers, and generalized derivations.

Let R be a ring and T: $R \to R$ be a left Jordan centralizer.

We define a function h: $R \times R \to R$ by

$$h(x, y) = T(xy) - T(x)y.$$

It is immediate that T is a (left) centralizer $\Leftrightarrow h(x, y) = 0$, $\forall x, y \in R$.

The intent is to study Jordan centralizers on rings wich are not semiprime.

Let $N(R)$ be the (Baer lower) radical of R, i.e., $R/N(R)$ is semiprime.

We show that $h(R,R) \subset N(R)$.

We do not show that $T(N(R)) \subset N(R)$.

This lecture will focus on the T invariant ideal generated by $H = h(R, R)$.

We show that I is the union of a chain of nilpotent ideals.

This shows that $I \subset N(R)$.

This follows us to prove that T is well-defined on the cosets of R/I and that this map is a centralizer on R/I.

Throught this lecture, we use the letter P to represent the set of absolute left annihilators:

$$P = \{x \in R;\ xR = 0\}.$$

P is a two sided ideal of R.

T will be a left Jordan centralizer.

We will be using repeated applications of T. In particular:

$$T^0 = x; \quad T^{n+1}(x) = T(T^n(x)).$$

The $\circ$ product or the *symmetric product* is defined by

$$x \circ y = xy + yx.$$

Lemma 1: Let T be a (left) Jordan centralizer on a ring R. Then

$$\sum_{i+j=n} T^i(x)T^j(x) = \sum_{i+j=n} T^i(xT^j(x)).$$

Lemma 2: Let R be a ring and T a Jordan centralizer on R. The function h is defined by $h = T(xy) - T(x)y$. The function h satisfies the following:

(i). $h(x,y) + h(y,x) = 0$;

(ii). $2h(xy,z) = -h(x,y)z + h(y,z)x - h(z,x)y$;

(iii). $h([x,y],z) = -h(x,y)z$;

(iv). $h(x,y)[z,w] + h(z,w)[x,y] = 0$,

$\forall x,y,z,w \in R$.

Lemma 3: More properties of h.

(i). $h(xy, z) = -h(xz, y)$;

(ii). $h(xy, z) = +h(y, xz)$;

(iii). $h([x, y]z, w) = 0$,

$\forall x, y, z, w \in R$.

Lemma 4: More properties of h.

(i). $h(x, y)[z, w]R = 0$;

(ii). $T^n(h(x, y))[z, w]R = 0$;

(iii). $h(x, y)T^n(h(z, w)) \subset T(P) + P$,

$\forall x, y, z, w \in R$.

An extremly important result follows from Lemma 4 (ii).

Suppose we have a product of the following form.

It has a $T^n(h(x, y))$ on the left end.

It ends with an element w of R on the right end.

The product of the elements $z_1 z_2 \ldots z_n$ which are sandwiched between the left end and the right end is independent of the order of the $z_1 z_2 \ldots z_n$.

$$T^n(h(x, y))z_1 z_2 \ldots z_n w = T^n(h(x, y))z_{i_1} z_{i_2} \ldots z_{i_n} w,$$

for any permutation $(i_1 i_2 i_3 \ldots i_n)$ of $(123 \ldots n)$.

A repeated use of this is to replace terms of the form

$$\ldots 2T^i(x)T^i(y)\ldots,$$

with

$$\ldots T^i(x{+}y)T^i(x{+}y)\ldots{-}\ldots T^i(x)T^i(x)\ldots{-}\ldots T^i(y)T^i(y)\ldots.$$

This allows us to reduce the problem to the case when the arguments of the two adjacent elements are equal.

This is necessary when we apply Lemma 1.

We shall refer to the elements which are sandwiched between $T^n(h(x,y))$ on the left and an element of R on the right as a *sandwiched string*.

Remember that the order of the elements of the sandwiched string is immaterial.

Lemma 5: Using the $\circ$ to denote the *symmetric product* $x \circ y = xy + yx$.

(i). $(h(R,R)) \circ h(R,R))R = 0$;

(ii). $h(x,y)h(z,w)h(u,v)R = 0$,

$\forall x, y, z, w, u, v \in R$.

Lemma 6: Let h_1 and h_2 be elements of $H = h(R,R)$. Then $T^n(h_1) \circ T^n(h_2)$ is equal to a linear combination of terms of the form

$$T^i(H)T^j(H) \text{ or } T^k(P),$$

where $i + j = 2n$, $i \neq j$ and $0 \leq k \leq 2n+1$.

Remember that P is the set of absolute left zero divisors of R.

In particular we can reduce the exponent to be less than n for one of the factors with the addition of terms involving $T^k(P)$.

Theorem 7: Any sandwiched product involving at least 2^n terms of the form $T^i(P)$ with $0 \leq i \leq n$ is zero.

Theorem 8: Any sandwiched product of enough terms of the form $T^i(H)$ with i bounded is zero.

Proof. The general idea of the proof is to use Lemma 6 to introduce a significant number of instances of $T^i(P)$ into eqch sandwiched string. Then the string evaluates to zero by Theorem 7. The total lenght involved will be two more than the total length of the sandwiched string to account for the left and right ends:
$2 + 3 \times 2^n \times 2^{2n+1} = 2 + 3 \times 2^{3n+1}$. ∎

Main Results and Applications

Theorem 9: Let T be a (left) Jordan centralizer on a ring R.
Let $h(x, y) = T(xy) - T(x)y$, $\forall x, y \in R$.
Let I be the T-invariant ideal generated by $h(R, R)$. Then I is the union of nilpotent ideals. T is well-defined on R/I. And T is a centralizer on R/I.

Proof. We use the notation $\langle X \rangle$ for the ideal of R generated by the set X.
Let $H_0 = \langle h(R, R) \rangle$.
Inductively, define $H_{n+1} = H_n + \langle T(H_n) \rangle$.
It is clear that each H_n is an ideal, and by Theorem 8, each H_n is nilpotent of index $\leq 2 + 3 \times 2^{3n+1}$.
Letting $I = \cup_{i=0}^{\infty} H_i$, I will be an invariant because $T(H_n) \subset H_{n+1}$.
I is a radical ring because it is the union of nilpotent ideals.
T is well-defined on R/I because $T(I) \subset I$.
T is a centralizer on R/I because
$h(x, y) \subset H_0 \subset I$. ∎

Remark: If R is a semiprime ring, then we can give a short proof for Zalar's result. We can also prove that any generalized Jordan derivation is a generalized derivation.

Zalar, B.: *On centralizers of semiprime rings*. Comment. Math. Univ. Carolinae. vol. 32, no. 4, 1991, 609-614.

Let R be a semiprime ring and T: $R \to R$ an additive mapping such that $T(x^2) = T(x)x$ holds for all $x \in R$. Then T is a left centralizer of R. It is also proved that Jordan centralizers and centralizers of R coincide.

Theorem 10: Let R be a 2-torsion-free semiprime ring and T: $R \to R$ a left Jordan centralizer, i.e., T is an additive mapping and satisfy $T(x^2) = T(x)x$, for all $x \in R$. Then T is a left centralizer, i.e., $T(xy) = T(x)y$, for all $x, y \in R$.

Corollary 11: Let R be a 2-torsion-free semiprime ring and G: $R \to R$ be a Jordan generalized derivation, i.e., G is an additive mapping satisfying the relation $G(x^2) = G(x)x + xD(x)$, for all $x \in R$ and some derivations D of R. Then G is a generalized derivation, i.e., G satisfies the relation $G(xy) = G(x)y + xD(y)$, for all $x, y \in R$ and some derivation D of R.

Proof. If $G(x^2) = G(x)x + xD(x)$, where D is a derivation, then $G - D$ is a left Jordan centralizer. So $G - D$ is a left centralizer, by the above Theorem, and so G is a generalized derivation. ∎

That’s my say!

Thank you for the superhuman patience!

Claus Haetinger

Research, Extension and Graduation Provost

Centro Universitário UNIVATES

95900-000, Lajeado-RS, Brazil

e-mail: chaet@univates.br

URL: http://ensino.univates.br/~chaet

Thank you for the “slaps”!