On Lie Structure of Prime Rings with Generalized (α, β)-Derivations

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Outline

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Acknowledgments

- Organizers;
- Co-authors;
- Prof. S.K. Nauman and Prof. M. Ashraf;
- FAPERGS: APE;
- UNIVATES: PPGAD, PPGECE, CETEC;
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Definitions and Notations

Prime Ring and Notations

- R associative ring with center $Z(R)$.
- $\forall x, y \in R$, $[x, y] = xy - yx$.
- $\forall x, y \in R$, $x \circ y = xy + yx$.
- R is prime if for any $a, b \in R$, $aRb = \{0\} \Rightarrow a = 0$ or $b = 0$.

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Lie Ideal

- L additive subgroup of R is a Lie ideal of R if $[L, R] \subseteq L$.
- L Lie ideal of R is square-closed if $x^2 \in L, \forall x \in L$.
- Let α and β be endomorphisms of R. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$.

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Generalized Derivations

- An additive mapping $F\colon R \to R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.
- Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers.
- Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d a derivation of R, is a generalized derivation; and if R has multiplicative identity 1, then all generalized derivations have this form.

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(α, β)-Derivations

- An additive map $d\colon R \to R$ is called an (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$.
- An $(1, 1)$-derivation is called simply a derivation, where 1 is the identity map on R.
- For a fixed a, the map $d_a\colon R \to R$ given by $d_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$ is an (α, β)-derivation which is said to be an (α, β)-inner derivation.
- An additive mapping $F\colon R \to R$ is called a generalized (α, β)-inner derivation if $F(x) = a\alpha(x) + \beta(x)b$, for some fixed $a, b \in R$ and for all $x \in R$.

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- An additive mapping $F \colon R \to R$ is called a generalized (α, β)-inner derivation if $F(x) = a\alpha(x) + \beta(x)b$, for some fixed $a, b \in R$ and for all $x \in R$.

Generalized (α, β)-Derivation

Definition

- A simple computation yields that if F is a generalized (α, β)-inner derivation, then for all $x, y \in R$, we have $F(xy) = F(x)\alpha(y) + \beta(x)d_{-b}(y)$, where d_{-b} is an (α, β)-inner derivation.
- With this viewpoint, an additive map $F\colon R \to R$ is called a generalized (α, β)-derivation associated with an (α, β)-derivation $d\colon R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$.
- An $(1, 1)$-generalized derivation is called simply a generalized derivation.

Outline

Motivation

- Over the last three decades, several authors have proved commutativity theorems for prime or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on some appropriate subsets of R (see Ashraf, Bell, Daif, Deng, Marubayashi and Rehman, where further references can be found).

Historical Account

- Ashraf and Rehman (2002): if a 2-torsion free prime ring R admits a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in L$, where L is a Lie ideal of R, then $L \subseteq Z(R)$.
- Rehman (2002) extended the mention results for generalized derivations.
- Our aim is to prove some results which are independent interest and related to generalized (α, β)-derivation on prime rings (BSPM, 2009).
- Our theorems extend some known theorems in the setting of generalized (α, β)-derivation in certain classes of rings.
- We refer the reader to Ashraf, Ali and Haetinger (ABM, 2006) and (IJMGTA, 2011) for an historical account and applications on derivations and higher derivations.

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Outline

Usual Identities

- We shall use without explicit mention the following basic identities, that hold for any $x, y, z \in R$:
- $[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y$;
- $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
- $(x \circ (yz))_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;
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- $((xy) \circ z)_{\alpha,\beta} = x(y \circ z)_{\alpha,\beta} - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)]$.

Preliminary Lemmas

Lemma (Bergen, Herstein, Kerr - J. Algebra, 1981)

If $L \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion-free prime ring R and $a, b \in R$ are such that $aLb = 0$, then $a = 0$ or $b = 0$.

Lemma (Rehman - Math. J. Okayama Univ., 2002)

Let R be a 2-torsion-free prime ring and L be a nonzero Lie ideal of R. If L is a commutative Lie ideal of R, then $L \subseteq Z(R)$.

3 Essentially Lemmas

Marubayashi, Ashraf, Rehman, Ali - Alg. Colloquium (to appear)

Lemma

Let R be a prime ring with $char(R) \neq 2$, and let L be a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R. If $[x, y]_{\alpha,\beta} = 0$, for all $x, y \in L$, then $L \subseteq Z(R)$.

Lemma

Let R be a 2-torsion-free prime ring and L be a nonzero square-closed Lie ideal of R. Suppose there exists a nonzero (α, β)-derivation d such that $d(x) = 0$ for all $x \in L$. Then $L \subseteq Z(R)$.

3 Essentially Lemmas

Marubayashi, Ashraf, Rehman, Ali - Alg. Colloquium (to appear)

Lemma

Let R be a 2-torsion-free prime ring and L be a nonzero square-closed Lie ideal of R. Suppose that α, β are automorphisms of R. If R admits a generalized (α, β)-derivation F with an associated nonzero (α, β)-derivation d such that $[F(x), x]_{\alpha,\beta} = 0$, for all $x \in L$, then $L \subseteq Z(R)$.

Outline

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.1)

Let R be prime ring, $char(R) \neq 2$ and L be a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R, and suppose R admits a generalized (α, β)-derivation F associated with an (α, β)-derivation d such that

(i), $F([x, y]) = (x \circ y)_{\alpha,\beta}$, *for all* $x, y \in L$, *or*

(ii), $F([x, y]) = -(x \circ y)_{\alpha,\beta}$, *for all* $x, y \in L$.

If $F = 0$ *or* $d \neq 0$, *then* $L \subseteq Z(R)$.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.2)

Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms of R, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that

(i), $F(x \circ y) = [x, y]_{\alpha,\beta}$ *for all* $x, y \in L$, *or*

(ii), $F(x \circ y) = -[x, y]_{\alpha,\beta}$ *for all* $x, y \in L$.

If $F = 0$ *or* $d \neq 0$, *then* $L \subseteq Z(R)$.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.3)

Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that

(i), $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ *for all $x, y \in L$, or*

(ii), $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$ *for all $x, y \in L$.*

Then $L \subseteq Z(R)$.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.4)

Let R be a prime ring, $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that

(i), $F([x, y]) = [F(x), y]_{\alpha,\beta}$ *for all* $x, y \in L$, *or*

(ii), $F([x, y]) = -[F(x), y]_{\alpha,\beta}$ *for all* $x, y \in L$.

Then $L \subseteq Z(R)$.

- If the commutator is replaced by the anti-commutator in the Theorem 3.4 , then we see that the conclusion of this theorem holds good.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.5)

Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that

(i), $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ *for all* $x, y \in L$, *or*

(ii), $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$ *for all* $x, y \in L$.

Then $L \subseteq Z(R)$.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.6)

Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively such that

(i), $F([x, y]) = [\alpha(y), G(x)]$ *for all* $x, y \in L$, *or*

(ii), $F([x, y]) = -[\alpha(y), G(x)]$ *for all* $x, y \in L$.

If $G = 0$ *or* $g \neq 0$, *then* $L \subseteq Z(R)$.

7 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th.3.7)

Let R be a prime ring with $char(R) \neq 2$ and L a nonzero square-closed Lie ideal of R. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively such that

(i), $F(x \circ y) = (\alpha(y) \circ G(x))$ *for all* $x, y \in L$, *or*

(ii), $F(x \circ y) = -(\alpha(y) \circ G(x))$ *for all* $x, y \in L$.

If $G = 0$ *or* $g \neq 0$, *then* $L \subseteq Z(R)$.

- In view of these results we get the following corollaries:

2 Corollaries

Corollary (R., Al-O., H. - BSPM, Cor. 3.7A)

Let R be a prime ring and I be a nonzero ideal of R. Suppose that α, β are automorphisms of R and R admits a generalized (α, β)-derivation F associated with an (α, β)-derivation d such that $\forall x, y \in I$

(i), $F([x, y]) = (x \circ y)_{\alpha,\beta}$ or $F([x, y]) = -(x \circ y)_{\alpha,\beta}$,

(ii), $F(x \circ y) = [x, y]_{\alpha,\beta}$ or $F(x \circ y) = -[x, y]_{\alpha,\beta}$,

(iii), $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ or $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$,

(iv), $F([x, y]) = [F(x), y]_{\alpha,\beta}$ or $F([x, y]) = -[F(x), y]_{\alpha,\beta}$,

(v), $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ or $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$.

If $F = 0$ or $d \neq 0$, then R is commutative.

2 Corollaries

Corollary (R., Al-O., H. - BSPM, Cor. 3.7B)

Let R be a prime ring and I be a nonzero ideal of R. Suppose that α, β are automorphisms of R and R admits generalized (α, β)-derivations F and G associated with (α, β)-derivations $d \neq 0$ and g respectively such that

(i), $F([x, y]) = [\alpha(y), G(x)]$, *or* $F([x, y]) = -[\alpha(y), G(x)]$ *for all* $x, y \in I$,

(ii), $F(x \circ y) = (\alpha(y) \circ G(x))$, *or* $F(x \circ y) = -(\alpha(y) \circ G(x))$ *for all* $x, y \in I$.

If $G = 0$ *or* $g \neq 0$, *then R is commutative.*

Outline

Morita Context

- Throughout this section we assume that the datum $K(R, S) = \{R, S, M, N, \mu_R, \tau_R\}$ is said to be Morita context (or MC) in which R and S are rings, M and N are (S, R) and (R, S)-bimodules respectively, $\mu_R : N \otimes_S M \to R$ and $\tau_S : M \otimes_R N \to S$ are bimodule homomorphisms with associative conditions $m_1 \mu_R(n \otimes m) = \tau_S(m_1 \otimes n)m$ and $\mu_R(n \otimes m)n_1 = n\tau_S(m_1 \otimes n_1)$, where μ_R and τ_S are called Morita's maps (or MC maps).
- If both Morita's maps are epimorphism then $K(R, S)$ is said to be a projective Morita context (or PMC).
- If one of the MC maps is an epimorphism, then $K(R, S)$ is said to be a semi-projective Morita context or semi-PMC.

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Morita Context with Derivations

- A classical problem in ring theory is to study and generalize conditions under which a ring becomes commutative.
- So far the best tools found for this purpose are the derivations on rings on their modules.
- One can also achieve this goal by comparing two rings and impose conditions on them.
- If one of the rings is appeared to be commutative, in compatible way, the other ring will also become commutative.
- In order to explore these ideas Morita theory is found to be a suitable tool.

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Muthana and Nauman Lemmas - E.W.J.M., 2008

Lemma (M-N, Th. 2.1)

Let R and S be rings of semi-PMC $K(R, S)$ in which τ_S is epic. If R is commutative and S is reduced, then S is also commutative.

Lemma (M-N, Cor. 2.4)

Let $K(R, S)$ be a PMC of rings in which R is commutative. Then

(i). *If S is a reduced ring, the R is also reduced and $R \cong S$.*

(ii). *If S is a domain, then both R and S become isomorphic integral domains.*

(iii). *If S is a division ring, then both R and S are isomorphic fields.*

3 Main Theorems

Theorem (R., Al-O., H. - BSPM, Th. 4.1)

Let R be prime ring, I a nonzero ideal of R and $K(R, S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits a generalized (α, β)-derivation F with an associated (α, β)-derivation d such that $F([x, y]) = (x \circ y)_{\alpha,\beta}$ or $F([x, y]) = -(x \circ y)_{\alpha,\beta}$, for all $x, y \in I$. If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.

Theorem (R., Al-O., H. - BSPM, Th. 4.2)

Let R be prime ring, I a nonzero ideal of R and $K(R,S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits a generalized (α,β)-derivation F with an associated (α,β)-derivation d satisfying any one of the following properties:

(i). $F(x \circ y) = [x, y]_{\alpha,\beta}$ or $F(x \circ y) = -[x, y]_{\alpha,\beta}$;

(ii). $[F(x), y]_{\alpha,\beta} = (F(x) \circ y)_{\alpha,\beta}$ or $[F(x), y]_{\alpha,\beta} = -(F(x) \circ y)_{\alpha,\beta}$;

(iii). $F([x, y]) = [F(x), y]_{\alpha,\beta}$ or $F([x, y]) = -[F(x), y]_{\alpha,\beta}$;

(iv). $F(x \circ y) = (F(x) \circ y)_{\alpha,\beta}$ or $F(x \circ y) = -(F(x) \circ y)_{\alpha,\beta}$.

If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.

Theorem (R., Al-O., H. - BSPM, Th. 4.3)

Let R be prime ring, I a nonzero ideal of R and $K(R, S)$ be a semi-PMC in which τ_S is epic. Let α, β be automorphisms, and suppose R admits generalized (α, β)-derivations F and G with associated (α, β)-derivations $d \neq 0$ and g respectively satisfying any one of the following properties:

(i). $F([x, y]) = [\alpha(y), G(x)]$, *or* $F([x, y]) = -[\alpha(y), G(x)]$ *for all* $x, y \in L$,

(ii). $F(x \circ y) = (\alpha(y) \circ G(x))$, *or* $F(x \circ y) = -(\alpha(y) \circ G(x))$ *for all* $x, y \in L$.

If S is reduced, then S is commutative. Moreover, if R and S are Morita similar rings, and S is a division ring, then R and S are isomorphic fields.

Outline

Later Works

- Nofayee and Nauman: Derivations on Morita rings and generalized derivations. WSPC (2010)
- Rehman, Al-Omary and Al-Shomrani: Morita context and generalized (α, β)-derivations (submitted);
- Nauman, Rehman and Al-Omary: Lie ideals, Morita context and generalized (α, β)-derivations (submitted).

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For Further Reading I

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International Journal of Mathematics, Game Theory and Algebra 19(5-6) (2011), pp. 359-379.

M. Ashraf; S. Ali; C. Haetinger.
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I think that's my say!

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