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ON DERIVATIONS IN RINGS AND THEIR APPLICATIONS

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Abstract. In this survey paper, we present an historical account on derivations, (θ,ϕ)-derivations, Jordan derivations, generalized derivations, generalized (θ,ϕ)-derivations, generalized Jordan derivations and other kinds of derivations in rings, based on the work of several authors. Moreover, recent results as well as some possible directions for future researches on the subject has been discussed in details. Finally, some applications of derivations have been given.

1. Introduction

Ring theory is a showpiece of mathematical unification, bringing together several branches of the subject and creating a powerful machine for the study of problems of considerable historical and mathematical importance. Rings with derivations are not the kind of subject that undergoes tremendous revolutions. However, this has been studied by many authors in the last 50 years, specially the relationships between derivations and the structure of rings.

One of the natural questions which often appeared in algebra and analysis is whether a map can be defined by its "local" properties. For example, the question whether a map, which acts like a derivation on the Lie product of some important Lie subalgebra of prime rings, is induced by an ordinary derivation, was a well-known problem posed by Herstein [112]. The first result in this direction was obtained in unpublished work of Kaplansky (cf. Herstein [112], p. 529), who considered matrix algebras over a field. With the presence of idempotent, this question has been examined by Martindale [168] for primitive rings. Herstein's problem was solved in full generality only after the powerful technique of functional identities was developed (see for example; [25], [27], [30], [54], where further references can be found). In the year 1993, Brešar [48] solved this problem for prime rings. Further, Beidar & Chebotar [28] solved this problem for Lie ideals of prime rings. The problem whether a Lie derivation is induced by an ordinary one related questions were also discussed in analysis viz. Banning & Mathieu [23], Villena [220], where further details can be looked.

This paper is an attempt to present the derivations and its variants in such a light, and in a manner suitable for everybody who have some basic knowledge in ring theory. In order to make the treatment as self-contained as possible, and to bring together all the relevant material in a single paper, we have included several references. Much of the motivation for this paper is historical, and we have taken the opportunity to weave historical comments into the body of the text where it seems appropriate.

Throughout the discussion, unless otherwise mentioned, R denotes an associative ring having at least two elements, with extended centroid C and symmetric quotient ring Q (see Beidar, et al. [30]). However, R may not have unity. The symbol $Z(R)$ stand for the center of R. Recall that a ring R is said to be *prime* if the product of any two nonzero ideals of R is nonzero. Equivalently, $aRb = \{0\}$ with $a, b \in R$ implies

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$a = 0$ or $b = 0$. A ring R is called *semiprime* if it has no nonzero nilpotent ideals. Equivalently, $aRa = \{0\}$ with $a \in R$ implies $a = 0$. For any $x, y \in R$, using its associative product one can induce two new products viz. the *Lie product* $[x, y] = xy - yx$ and the *Jordan product* $x \circ y = xy + yx$. An additive subgroup $U \subset R$ is said to be a *Lie ideal* (resp. *Jordan ideal*) of R if whenever $u \in U$ and $r \in R$, then $[u, r]$ (resp. $(u \circ r)$) is also in U. Let S be a nonempty subset of R. A function f: $R \longrightarrow R$ is said to be a *centralizing function* on S if $[f(x), x] \in Z(R)$, for all $x \in S$. In the special case if $[f(x), x] = 0$, for all $x \in S$, f is said to be commuting on S.

A map d: $R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies the Leibnitz' rule; $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$. A simple example is of course the usual derivative on various algebras consisting of differentiable functions. Basic examples in noncommutative rings are quite different. Note that $[a, xy] = [a, x]y + x[a, y]$, for all $a, x, y \in R$. For a fixed $a \in R$, define d: $R \longrightarrow R$ by $d(x) = [x, a]$ for all $x \in R$. The function d so defined can be easily checked to be additive and

$$d(xy) = [xy, a] \ = x[y, a] + [x, a]y = xd(y) + d(x)y, \text{ for all } x, y \in R.$$

Thus, d is a derivation which is called *inner derivation* of R associated with a and is generally denoted by I_a. It is obvious to see that every inner derivation on a ring R is a derivation. But one can find plenty of examples of derivations which are not inner.

If R is a commutative ring with identity 1 and d a derivation of R, then a skew polynomial ring $R[x; d]$ is defined as the set S of all polynomials $\sum_{i=0}^{n} r_i x^i$ with usual addition and the multiplication by the rule $xr = rx + d(r)$, for all $r \in R$. A derivation d of R is said to be X-*inner* if there exists $a \in Q$ such that $d(x) = [a, x]$, for all $x \in R$. Derivations that are not X-inner are called X-*outer*. Denote by $Der(R)$, the set of all derivations of R and let $Inn(R) = \{d \in Der(R)C \mid d = ad(A), \text{ for some } A \in Q\}$. Elements of $Inn(R)$ are called X-inner derivations and other elements of $Der(R)$ are called outer derivations. Assume that R is an algebra over the rational field $\mathbb{Q}$ and d: $R \longrightarrow R$ is a derivation. Then, if we put $d_n(x) = \frac{d^n(x)}{n!}$, for every $n \in \mathbb{N}$, we have that

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b), \text{ for all } a, b \in R \text{ and, } n \geq 1. \tag{1.1}$$

So d defines a sequence $d_0, d_1, \ldots, d_n, \ldots$ such that $d_0 = id_R$, d_1 is a derivation and equation (1.1) holds. A sequence of additive mappings $D = \{d_0, d_1, \ldots, d_n, \ldots\}$ is said to be a *higher derivation* of R if the above relation (1.1) holds ([128], Exerc. 4, p. 540). More precisely, higher derivation in a ring R is a sequence of additive mappings $D = (d_i)_{i \in \mathbb{N}}$ of R satisfying the conditions $d_0 = id_R$ and $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in R$ and for all $n \in \mathbb{N}$. For details study and examples we refer to readers ([90], [91], [104]).

Many questions on derivations have been considered during the development of the theory. For example:

- powers (or products) of derivations and commutativity of rings (Posner's theorems) were considered in ([1], [20], [37], [67], [79], [84], [123], [133], [148], [149], [152], [154], [171], [172], [196], [208], [224]);
- algebraic derivations were considered in ([3], [21], [32], [40], [56], [72], [86], [142], [143], [147], [148], [150], [153], [164], [170], [194]);
- the relationship between the associative, the Jordan and Lie structure of associative rings (Herstein's question) were considered in ([95], [98], [104], [108], [111], [112], [114], [180]);
- integral derivations were considered in ([3], [87], [94], [95], [190], [202], [203], [204]);
- derivations in many types of rings were considered in ([5], [21], [24], [25], [26], [42], [65], [70], [71], [76], [77], [93], [150], [161], [166], [167], [168], [170], [210], [220], [230], [233], [234]).

They were already generalized in several directions.

Finally, let us say that the historical approach of this paper is partially based on work of several authors, while the most of the sections were strongly based on the authors background and researches. We present some recent applications of derivations, as well as some problems that could be searched. We do not have the pretension to list here all the possible results and problems, neither to affirm that the cited results are the most important. One more time, the interested readers can consult the innumerable references cited here.

2. Historical Note

Following Nowicki [190], the fundamental relations between the operation of differentiation (=derivation) and that of addition and multiplication of functions have been known for a long time as the notion of the derivative itself. The relations were deepended when it was found that the operation of differentiation of functions on the smooth varieties with respect to a given tangent field not only has the formal properties of differentiation but also conversely; the tangent field as fully characterized by such an operation. Therefore, it was possible to define e.g. the tangent bundle in terms of sheaves of functions.

The notion of the ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In the 1940's it was found that the Galois theory of algebraic equations can be transferred to the theory of ordinary linear differential equations (the Picard-Vessiot theory, including Picard-Vessiot theories for differential equations and for difference equations). In the usual sense, "Picard-Vessiot theory" means a Galois theory for linear ordinary differential equations (cf. Van der Put & Singer [217] for details). The filed theory also included the derivations in its inventory of tools. The classical operation of differentiation of forms on varieties led to the notion of differentiation of singular chains on varieties, a fundamental notion of the topological and algebraic theory of homology.

In the 1950's a new part of algebra called differential algebra was initiated by the works of Ritt & Kolchin. In 1950, Ritt [197] and in 1973, Kolchin [143] wrote the well-known books on differential algebra. Kaplansky [139] also wrote an interesting book on this subject in 1976.

3. Chronological Development

The study of derivations in rings though initiated long back, but got impetus only after Posner [195] who in 1957 stablished two very striking results on derivations in prime rings. The result under reference state that; (*i*). *In a 2-torsion-free prime ring, if the iterate of two derivations is a derivation, then one of them must be zero;* (*ii*). *A prime ring R admitting a nonzero centralizing derivation d must be commutative.* The notion of derivation has also been generalized in various directions such as Jordan derivation, left derivation, (θ, ϕ)-derivation, generalized derivation, generalized Jordan derivation, generalized Jordan (θ, ϕ)-derivation, higher derivations, generalized higher derivations, etcetera. Also, there has been considerable interest in investigating commutativity of rings, more often that of prime and semiprime rings admitting these mappings which are centralizing or commuting on some appropriate subsets of R. Being important ring theory tools (see for example [37]), these results are one of the sources of the development of such as the theory of differential identities (see [143]), theory of Hopf algebra action on rings (see [179], [213]) and Galois theory for linear ordinary differential equations (cf.; Van der Put & Singer [217]).

3.1. Posner's Theorems

The specific statements of Posner's theorems, to which we shall referer frequently, are the following:

Posner's First Theorem. Let R be a prime ring of characteristic not 2 and d_1, d_2 be derivations on R such that the iterate $d_1 d_2$ be also a derivation, then one at least of d_1, d_2 is zero.

Posner's First Theorem tells us that the composition of two nonzero derivations of a prime ring R can not be a derivation provided that characteristic of R is different from 2. Thereafter, a number of authors have generalized this theorem in several ways (see for example Bergen [37], Chebotar [66], Chuang [68], [69], Hirano et al. [120], Hvala [127], Jensen [133], Krempa [147], Lanski [154], Martindale [170] and Ye et al. [229] where further references can be found).

Generally speaking, a composition of inner derivations can be a nonzero derivation. For example; if e is a nonzero idempotent in R i.e., $e^2 = e \neq 0$, then $(ad(e))^{2k-1} = ad(e)$ for any positive integer k (see in Lanski [155] for more interesting example). If d is any derivation of the prime ring R, then d^p is a derivation of R provided that char$(R) = p$. However, it was not clear in general whether a composition of derivations could be a nonzero derivation if some of them are inner and some of them are outer. Some progress was achieved by applying result of Kharchenko [143] on independence of outer derivations. The result on composition of three derivations was obtained by Lanski [155] in 1992 as follows:

Theorem 3.1.1. Let R be prime ring of characteristic different from 2 and $d_1, d_2, D \in Der(R)/\{0\}$ so that $d_1 d_2 D = E \in Der(R)$. Then either $d_1, d_2, D \in Inn(R)$, or else char$(R) = 3$, d_1 is outer, $d_2 = d_1 z_1, D = d_1 z_2$ and $(z_1)^{d_1} = 0$, where $z_i \in C$, so $E = {d_1}^3 z_1 z_2$.

In the same paper, Lanski posed the question whether a composition of fewer than char(R) derivations, or any product in case char$(R) = 0$ be a nonzero derivation if some are inner and some are outer. Further, in 1995 Chebotar [66] obtained the necessary condition when the composition of derivations, including both inner and outer ones, could be a derivation (see Beidar et al [30] for more details). However, Lanski [155] question remained open till date. Very recently, Chebotar & Lee [67] present the partial answer of Lanski question by means of following example:

Example 3.1.1. Let F be a field of characteristic different from 2 and let $R = M_2(F[x])$ be the ring of 2×2 matrices over the ring of polynomials in indeterminate x over F. Take $d \in Der(R)$ defined by applying formal differentiation to each entry. Set $d_1 = d_3 = d_5 = ad(e_{12}), d_2 = d + ad(e_{21}), d_4 = d - ad(e_{21})$. Then d_2 and d_4 are outer derivations and $d_1 d_2 d_3 d_4 d_5 = ad(-4e_{12})$.

Moreover, Posner's First Theorem was rediscovered by Creedon [79] to semiprime algebras. In fact, he proved that if the product of two derivations in an algebra A is a derivation, then the product maps the algebra into the nil radical $nil(A)$ (the intersection of all prime ideals of A). Thus, if the product of two derivations in a semiprime algebra is a derivation, then the product is zero. Further, Creedon obtained conditions proving that the product of two derivations maps the algebra into the Jacobson radical ([79], Proposition 9).

Many authors have investigated the invariance of certain ideals under derivations. It is known that bounded derivations on Banach algebras leave primitive ideals invariant [208] and derivations on characteristic-free rings leave minimal prime ideals invariant ([84], 3.3.2). Creedon showed that if P is a prime ideal of a ring R, where the characteristic of R/P is not two, such that the product of two derivations leaves P invariant, then one of the derivations must leave P invariant. He also proved that, if d is a derivation on a ring R and P is a semiprime ideal of R, such that R/P is characteristic-free and $d^k(P) \subseteq P$, for any fixed positive integer k, then $d(P) \subseteq P$. For more related results see e.g.; Bell [31], Bell & Argaç [32], Hirano et al. [111], Jensen [133], Krempa [148], Lanski [154] and Wang [224].

Posner's Second Theorem Let R be a prime ring. If there is a nonzero centralizing derivation of R, then R is commutative.

This theorem says that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Considering this theorem from some distance it is not entirely clear to us what was Posner's motivation for proving it and for which reasons he was able to conjecture that the theorem is true. Any how it is a fact that the theorem has been extremely influential and at least indirectly it initiated the study of commuting derivations i.e., the topic arising directly from Posner's Second Theorem. It should be mentioned that Posner in fact proved this theorem under the more general condition that d satisfies $[d(x), x] \in Z(R)$, for every $x \in R$. Maps satisfying this condition are usually called centralizing in the literature. It has turned out that under rather mild assumptions a centralizing map is necessarily commuting (see for example [45], Proposition 3.1).

Remark 3.1.1. It is evident by the following example that Posner's Second Theorem can not be extended for arbitrary rings. Consider a ring $R = R_1 \times R_2$, where R_1 and R_2 are nonzero rings. If R_1 is a commutative ring having a nonzero derivation d_1 and R_2 is a noncommutative ring, then R is a noncommutative ring and $d(x_1, x_2) = (d_1(x_1), 0)$ is a nonzero commuting derivation on R. However, R is not commutative. This is a trivial example, but it explains well why the assumption of primeness is natural in Posner's Second Theorem.

Over the last 50 years, a lot of work has been done on centralizing and commuting mappings. A number of authors have extended these results by considering mapping which are only assumed to be centralizing on an appropriate subset of the ring. In the year 1973, Awtar [19] considered centralizing derivations on Lie and Jordan ideals. In the Jordan case, he proved that if a prime ring of characteristic not two has a nontrivial derivation which is centralizing on a Jordan ideal, then the ideal must be central. More precisely, he obtained the following results:

Theorem 3.1.2 ([19], Theorem 1). Let R be a prime ring of characteristic different from 2 and 3. Let d be a nonzero derivation of R, and U a Lie ideal of R with $[u, d(u)] \in Z(R)$, for all $u \in U$. Then $U \subset Z(R)$.

Theorem 3.1.3 ([19], Theorem 2). Let R be a prime ring of characteristic 2, and let d be a nonzero derivation of R. Let U a Lie (Jordan) ideal and a subring of R. Suppose that $[u, d(u)] \in Z(R)$, for all $u \in U$. Then R is commutative.

It is to remark that in the hypotheses of Theorem 3.1.3, if we just assume that U is only a Lie (Jordan) ideal or a subring of R, then U may not be commutative. This is shown by the following examples due to Awtar [19].

Example 3.1.1. Let R be a prime ring of all 2×2 matrices over a noncommutative prime ring. Consider $U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R \right\}$. It is clear that U is a subring, but not a Lie ideal of R. Define a mapping d: $R \longrightarrow R$ as follows:

$$d\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}.$$

Then, it is easy to verify that d is a nonzero derivation of R with $[u, d(u)] \in Z(R)$, for all $u \in U$. But U is not commutative.

Example 3.1.2. Consider the prime ring R of all 2×2 matrices over $GF(2)$.
Let $U = \left\{ \begin{pmatrix} x & y \\ z & x \end{pmatrix} \mid x, y, z \in R \right\}$. It is clear that U is a Lie ideal, but not a subring of R. Let us define a map d: $R \longrightarrow R$ as follows: $d\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = \begin{pmatrix} w-z & x-w \\ x-w & y-z \end{pmatrix}$. Then, it is clear that d is a nonzero derivation of R such that $[u, d(u)] \in Z(R)$, for all $u \in U$. However, U is not commutative.

In 1976, Mayne [174] obtained the analogous result for an automorphism which states as follows:

Theorem 3.1.3 ([174], Theorem). If R is a prime ring with a nontrivial centralizing automorphism, then R is a commutative integral domain.

In the year 1982, Mayne [176] extended the above results and established that the underlying automorphism or derivation needs only to be centralizing and invariant on a nonzero ideal in order to ensure the commutativity of a prime ring. It was also shown that if the prime ring is of characteristic different from two, then the mapping needs only to be centralizing and invariant on a nonzero Jordan ideal. Later, in the year 1984, Mayne [175] pointed out that the ideal invariant assumption is unnecessary in the above theorem and he proved that the existence of a nontrivial automorphism or derivation which is centralizing on a nonzero ideal in a prime ring R implies that the ring R must be commutative. Then Mayne using the fact that every nonzero quadratic Jordan ideal contains a nonzero (associative) ideal [178], find that the mapping needs only be centralizing on a nonzero quadratic Jordan ideal. In the derivation case this extends Theorem 3 of Awtar [19] to prime rings of arbitrary characteristic. In 1969, McCrimmon [178] showed that in the automorphism case the results of Mayne (Theorem 2 of [175]) can not be extended to semiprime rings.

Recently, Vukman [223] extended Posner's Second Theorem by showing that if d is a derivation of prime ring of characteristic not 2 such that $[[d(x), x], x] = 0$, for all $x \in R$, then $d = 0$ or R is commutative. In fact, in the spirits of Posner's theorem, he merely showed that d is commuting. In 1992, the result proved in [175] was further generalized for automorphism or derivation centralizing on a nontrivial Lie ideal.

Theorem 3.1.4 ([174], Theorem). Let R be a prime ring of characteristic different from two and T be an automorphism of R which is centralizing and nontrivial on a Lie ideal U of R. Then U is contained in Z(R).

In 1993 Brešar [46] proved that same concrete additive mappings (such as derivation, endomorphism, etc.) can not be centralizing on certain subsets of noncommutative prime (and some other) rings. In the same paper, he also described the structure of an arbitrary additive mapping which is centralizing on a prime ring and proved the following result:

Theorem 3.1.5 ([46], Theorem A). Let R be a prime ring. Suppose an additive mapping F of R into itself is centralizing on R. If either R has a characteristic different from two or F is commuting on R, then F is of the form $F(x) = \lambda x + \zeta(x),\ x \in R$, where λ is an element from the extended centroid C of R and ζ is an additive mapping of R into C.

The proof of Theorem 3.1.5 depend on the identity $d(x) = ag(x) + h(x)b$, for all $x \in R$ (see [46], Theorem 2.1) and a, b are some fixed element of R, which gives a description of derivations d, g and h of prime ring R. Brešar in [46] initiated the study of a more general concept than centralizing mapping i.e., he consider the situation when the mappings F and G of a ring R satisfy $F(s)s - sG(s) \in Z(R)$, for all s in some subset S of R. In fact he proved the following theorem:

Theorem 3.1.6 ([46], Theorem B). Let R be a prime ring and U be a nonzero left ideal of R. Suppose derivations d and g of R satisfy $d(u)u - ug(u) \in Z(R)$, for all $u \in U$. If $d \neq 0$ then R is commutative.

This has been inspired by the following observation. Let f be a generalized inner derivation of R, i.e., $f(x) = ax + xb$, for some $a, b \in R$. Note that the condition that f is centralizing on subset S of R can be written in the form $[a, s]s - s[s, b] \in Z(R)$, for all $s \in S$. Thus, introducing inner derivation d and g by $d(x) = [a, x]$ and $g(x) = [x, b]$, we obtain the same condition as in Theorem 3.1.6, i.e., $d(s)s - sg(s) \in Z(R)$, for all $s \in S$. Generalized inner derivations are extensively studied on operator algebras. Therefore, it might be interesting to investigate these mappings from an algebraical piont of view also.

Numerous conditions concerning additive maps which are more general than f being centralizing, in particular commuting, but usually implying the same conclusion, have been studied by many algebraists. It would occupy too much space to discuss at greater length all of them, so we just refer the reader to some references viz; Awtar [18], [19], [20], Bell [31], Bell & Argaç [32], Bell & Martindale [34], Brešar [46], [47], [48], [51], [52], Brešar & Vukman [61], Hirano et al. [120], Hongan [123], Lanski [154], Luh [165], Mayne [173], [174], [176], [175], McCrimmon [178], Vukman [223] and Wong [224], where further reference can be found, for a state-of-art account and comprehensive bibliography.

3.2. Herstein's Problem

In 1950's, Herstein initiated the study of the relationship between the associative and the Jordan and Lie structure of associative rings. We refer the reader to ([112], [114], [115]), where one can find further references and more detailed explanations concerning the motivation and the background of these researches.

A *Jordan derivation* d of a ring R is an additive mapping d: $R \longrightarrow R$ such that $d(a^2) = d(a)a + ad(a)$, for every $a \in R$. Every derivation is obviously a Jordan derivation and the converse is in general not true.

Example 3.2.1. Let R be a 2-torsion-free ring and $a \in R$ such that $xax = 0$ for all $x \in R$, but $xay \neq 0$, for some $(x \neq y) \in R$. Define a map d: $R \longrightarrow R$ as follows: $d(x) = ax$. Then, it can be verified that d is a Jordan derivation but not a derivation.

One can verify that a Jordan derivation in associative ring R is a derivation on the Jordan ring under the induced Jordan multiplication. Note that the definition of Jordan derivation presented in the work of

Herstein is not as the given above. In fact, Herstein constructed, starting from the ring R, a new ring, namely the Jordan ring R, defining the product in this one as being $a \circ b = ab + ba$ for any $a, b \in R$. Clearly, this new product is well-defined and it can be easily verified that $(R, +, \circ)$ is a ring. So, an additive mapping d, from the Jordan ring into itself, is said by Herstein to be a Jordan derivation, if $d(a \circ b) = d(a) \circ b + a \circ d(b)$, for every $a, b \in R$. However, in the year 1957, Herstein proved a classical result in this direction which becomes a jumping point for many workers later. The result to which we refere is namely:

Theorem 3.2.1 ([111], Theorem 3.1). If R is a prime ring of characteristic different from 2, then every Jordan derivation of R is a derivation.

In the year 1988, Brešar & Vukman [58] presented a brief (alternative) proof of this classical result. If one checks the proof given in Theorem 3.2.1 one sees that the assumption that the characteristic of R be different from 2 enters only at two points; in proving $d(aba) = d(a)ba + ad(b)a + abd(a)$, for all $a, b \in R$ and at the very end of the argument just given. If we redefine a *Jordan derivation* by $d(a^2) = d(a)a + ad(a)$ and $d(aba) = d(a)ba + ad(b)a + abd(a)$, then in the ring of characteristic not 2 we have imposed no extra restriction yet in characteristic 2 it allows us to conclude:

Theorem 3.2.2 ([58], Theorem 3.4). If R is a prime ring and d is a Jordan derivation (as newly redefined) of R, then d is a derivation except if R is both commutative (and so an integral domain) and of characteristic 2.

Later on Brešar [49] extended the result to 2-torsion-free semiprime rings. In a subsequent paper, Brešar gave another proof of this result using Jordan triple derivations. An additive mapping $d\colon R \to R$ is said to be a *Jordan triple derivation* if $d(aba) = d(a)ba + ad(b)a + abd(a)$, for every $a, b \in R$. He proved that every Jordan triple derivation of a 2-torsion-free semiprime ring is a derivation ([50], Theorem 4.3). It turns out that every Jordan derivation of a 2-torsion-free ring is a Jordan triple derivation ([116], Lemma 3.5). This gives another proof of the result of Herstein for 2-torsion-free semiprime rings. Further, Awtar extended the Herstein's theorem to Lie ideals ([18], Theorem). He proved that if U is a Lie ideal of a prime ring R of characteristic different of 2 such that $u^2 \in U$, for every $u \in U$, and $d\colon R \to R$ is an additive mapping such that $d\,|_U$ is a Jordan derivation of U into R, then $d\,|_U$ is a derivation of U into R. In 2000, Ashraf & Rehman [14] proved that, if R is a 2-torsion-free prime ring and if U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$ (*square closed*) and $d\colon R \to R$ is an additive map satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then $d(uv) = ud(v) + vd(u)$, for all $u, v \in U$. An additive mapping $d\colon R \to R$ is called a *Jordan left derivation* if it satisfies the above property that is $d(x^2) = 2xd(x)$ for all $x \in R$. In 1990, Brešar & Vukman [60] have proved that the existence of a nonzero Jordan left derivation on a prime ring R of $char(R) \neq 2, 3$ forces R to be commutative. Later in 1992, Deng [83] improved the above result, proving that if R is a prime ring of characteristic $\neq 2$, X is a nonzero left R-module that is faithful and prime, and if there exists a nonzero Jordan left derivation $d\colon R \to X$, then R is commutative.

3.3. (θ, ϕ)-Derivations

Jacobson in his classical book "*Structure of Rings*" [132] has given a passing reference of (s_1, s_2)-derivation which was latter more commonly referred as (σ, τ)-derivation or (α, β)-derivation by some authors and (θ, ϕ)-derivation by others like Argaç et al. [7], Brešar & Vukman [59], Kaya [141], Yenigül et al [231], to mention a few only. Let θ, ϕ be endomorphisms of R. An additive mapping $d\colon R \longrightarrow R$ is called a *(θ, ϕ)-derivation* (resp. *Jordan (θ, ϕ)-derivation*) on R if $d(xy) = d(x)\theta(x) + \phi(x)d(y)$ holds for all $x, y \in R$ (resp. $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$ holds for all $x \in R$).

A mapping $a \mapsto \theta(a)b - b\phi(a)$, where b is a fixed element in R is a (θ, ϕ)-derivation. Such a (θ, ϕ)-derivation is said to be *inner*. A $(\theta, 1)$-derivation, where 1 is the identity map on R is called simply a *θ-derivation*. Of course, 1-derivation is a derivation. An additive mapping $\delta\colon R \longrightarrow R$ is called a *left (θ, ϕ)-derivation* (resp. *left Jordan (θ, ϕ)-derivation*) if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$, for all $x, y \in R$ (resp. $\delta(x^2) = \theta(x)\delta(x) + \phi(x)d(x)$, for all $x \in R$).

Leroy & Matczuk [164] generalized Herstein's [111] result to Jordan θ-derivations, where θ is an automorphism ([164], Theorem 2.6). Further, in the year 1991 Brešar & Vukman [59] extended Herstein's [111] result to Jordan (θ, ϕ)-derivations and proved the following:

Theorem 3.3.1. Let R be any ring and R' be a noncommutative ring. Let θ and ϕ be homomorphisms of R into R'. Let X be a 2-torsion-free R'-bimodule. Suppose that either θ is onto and $xR'a = 0$ with $x \in X,\ a \in R'$ implies that $x = 0$ or $a = 0$ or that θ is onto and $aR'x = 0$ with $x \in X,\ a \in R'$ implies that $x = 0$ or $a = 0$. In this case every Jordan (θ, ϕ)-derivation d: $R \longrightarrow X$ is a (θ, ϕ)-derivation.

Theorem 3.3.2. Let R be a commutative prime ring (i.e., a commutative integral domain) of characteristic different from two. If θ and ϕ are any endomorphisms of R, then every Jordan (θ, ϕ)-derivation d of R is a (θ, ϕ)-derivation. Moreover, if $\theta \neq \phi$, then there exists an element λ in the field of fractions F of R such that $d(a) = \lambda(\phi(a) - \theta(a))$, for all $a \in R$.

If R is a ring with involution $\star$, then every additive mapping E: $R \to R$ which satisfies $E(x^2) = E(x)x^\star + xE(x)$ for all $x \in R$ is called a *Jordan $\star$-derivation.* Following [233] these mappings are closely connected with a question of representability of quadratic forms by bilinear forms. In Theorem 2.1 of [62], Brešar & Zalar obtained a representation of Jordan $\star$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. In [233], Zalar proved that any left (resp. right) Jordan centralizer on a 2-torsion-free semiprime ring is a left (resp. right) centralizer. Cortes & Haetinger [77] proved this question changing the semiprimality condition on R. The main result of this paper is the following: Let R be a 2-torsion-free ring which has a commutator right (resp. left) nonzero divisor and let G: $R \to R$ be a left (resp. right) Jordan σ-centralizer mapping of R, where σ is an automorphism of R. Then G is a left (resp. right) σ-centralizer mapping of R.

In 2001, Ashraf et al. [15] considered the following problem: Let R be a prime ring, $char(R) \neq 2$, and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. They showed that if d is an additive mapping of R into itself satisfying $d(u^2) = 2ud(u)$, for all $u \in U$, then either $U \subseteq Z(R)$ or $d(U) = 0$. In 2005, Ashraf [8] proved, with the same assumption on R, θ, φ as above, that if R admits a nonzero left Jordan (θ, φ)-derivation, then R is commutative. Further, as an application of this result it was shown that every left Jordan (θ, φ)-derivation on R is a left (θ, φ)-derivation on R. Finally, in case of an arbitrary prime ring it was proved that if R admits a left (θ, φ)-derivation which acts also as a homomorphism (resp. anti-homomorphism) on a nonzero ideal of R, then $d = 0$ on R.

Remark 3.3.1. Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorem holds even if U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are not ideals. For example, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in Z \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in Z \right\}$ is a Lie ideal of R satisfying $u^2 \in U$, for all $u \in U$. However, U is not an ideal of R.

Here is a related open problem that awaits solution:

Open Problem 3.3.1. Let R be a 2-torsion-free prime ring and U be a nonzero Lie ideal of R. If R admits a left Jordan derivation δ such that $d(u^2) = 2ud(u)$, for all $u \in U$, then either $U \subseteq Z(R)$ or $d(U) = 0$.

Over the recent years, a number of authors have extended Herstein's theorems for semiprime rings, Lie ideals, superalgebras, local rings, and rings containing some special conditions. Moreover, these problems have been extended to many kinds of derivations viz.,

- Left derivations: ([2], [8], [14], [15], [60], [77], [232]);
- Jordan derivations: ([18], [19], [29], [49], [50], [60], [177]);
- Generalized derivations: ([2], [9], [13], [16], [134], [183], [184], [185]);
- Triple derivations: ([49], [133], [182], [183]);
- Higher derivations: ([90], [91], [104], [133], [146], [186], [229]);
- Super derivations: ([94], [97], [179]), where further references can be found.

3.4. Generalized Derivations

During the last few decades there has been a great deal of work concerning generalized derivation in context of algebras on certain normed spaces (for reference see [127], where further references can be found). By a generalized derivation on an algebra A, one usually means a map of the form $x \mapsto ax + xb$, where a and b are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e., the map of the form $x \mapsto ax - xb$). In the theory of operator algebras, they are considered as an important class of the so-called elementary operators, that is, operators where $x \mapsto \sum_{i=1}^{n} a_i x b_i$. Now in a ring R, let F be a generalized inner derivation given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_b(y)$ where $I_b(y) = yb - by$ is the inner derivation defined by $b \in R$. Motivated by this observation in the year 1991, Brešar [53] introduced the concept of generalized derivation in rings as follows:

Definition 3.4.1. Let S be a non-empty subset of R. An additive mapping F: $R \longrightarrow R$ is said to be a *generalized derivation* on S if there exists a derivation d: $R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in S$.

Recently, Hvala [127] initiated the algebraic study of generalized derivation, a function more general than derivation and extended some results concerning derivations to generalized derivations. In fact, the concept of generalized derivation covers both concept of derivation as well as that of generalized inner derivation. Moreover, generalized derivations with $d = 0$ covers the concept of left multipliers, that is, an additive map f satisfying $f(xy) = f(x)y$, for all $x, y \in R$. This has widely been studied in functional analysis and several interesting results are obtained (see, for example; Are & Mathiew [4], Sinclair [208], and Wendel [224], where further references can be found).

Let S be an algebra over a commutative ring R and M an S/R-bimodule. If M and N are S/R-bimodules, a homomorphism f: $M \longrightarrow N$ means a R-module and a two sided S-module map. A R-module map d: $S \longrightarrow M$ is called a derivation or inner derivation if $d(st) = d(s)t + sd(t)$ or if $d(s) = ms - sm$ for some $m \in M$, respectively $(s, t \in S)$. We denote the set of derivations (resp. inner derivations) from S to M by $Der_k(S, M)$ (resp. $Inn_k(S, M)$). $Der_k(S, M)$ is an R-module and $Inn_k(S, M)$ is an R-submodule of $Der_k(S, M)$. An R-module map f: $S \longrightarrow M$ is called a generalized derivation if there exists a derivation d: $S \longrightarrow M$ such that $f(st) = f(s)t + sd(t)$, for all $s, t \in S$ and for $m, n \in M$, a map $f_{m,n}$: $S \longrightarrow M$ such that $s \mapsto ms + sn \in M$ is called a generalized inner derivation. For an R-module map f: $S \longrightarrow M$ and an element $m \in M$, a pair (f, m) is called a generalized derivation if $f(st) = f(s)t + sf(t) + smt$, for any $s, t \in S$. And by $f_{m,n}(st) = f_{m,n}(s)t + sf_{n,m}(t) + s(-m-n)t$ is called generalized inner derivation and is denoted by $(f_{m,n}, -m-n)$. Two generalized derivations (f, m) and (g, n) are equal if $f = g$ and $m = n$. Under some conditions, m is uniquely determined by f. We also denote the set of generalized derivations (resp. generalized inner derivations) from S to M by $gDer_k(S, M)$ (resp. $gInn_k(S, M)$). In 1999, Nakajima [185] gave some elementary properties of generalized derivations defined by Brešar [53], and determined functorial relations between $gDer_k(S, M)$ and $Der_k(S, M)$. Using this result, he gave the universal mapping property of generalized derivations in the above sense. Some more related results can be looked in Komatsu & Nakajima [145], Nakajima [186] and Nakajima & Sapanci [187], where further reference can be found).

In the year 2001, Lee [158] extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive map F: $\varrho \longrightarrow U$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in \varrho$, where ϱ is a dense right ideal of R and d is a derivation from ϱ into U, right Utumi quotient ring. He proved that every generalized derivation can be uniquely extended to a generalized derivation of U. In fact, there exists $a \in U$ and a derivation d of U such that $F(x) = ax + d(x)$ for all $x \in U$ ([158], Theorem 3). Therefore we may assume without loss of generality that a generalized derivation of R is a map $U \longrightarrow U$. In [143], Kharchenko described identities with derivations and his results are powerful tool for reducing a differential identity to a generalized polynomial identity. Thus, to study identities with generalized derivations, it seems reasonable to find a corresponding theorem for identies with generalized derivations. In [160], Lee & Shinu proved that if $f(X_i{}^{\Gamma_j})$ is an identity for R, where the Γ_j's are distinct regular words in generalized derivations, then $f(Z_{ij})$ is a generalized polynomial identity (GPI) for U. They also obtained some results concerning identities with generalized derivations. In particular, they generalized Theorem 1 and 2 of [127] to prime rings without the characteristic assumption. Further, they prove an analogous theorem for prime rings with involution.

During the last decade, there has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. Recently, many authors viz [32], [35], [45] and [124] have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. In the year 2001, Ashraf & Nadeem [12] established that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying $d(xy)+xy \in Z(R)$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$. Motivated by these observations, Ali in [2] explore the commutativity of a ring R satisfying any one of the properties: (i). $F(xy) - xy \in Z(R)$, (ii). $F(xy) + xy \in Z(R)$, (iii). $F(xy) - yx \in Z(R)$, (iv). $F(xy) + yx \in Z(R)$, (v). $F(x)F(y) - xy \in Z(R)$ and (vi). $F(x)F(y) + xy \in Z(R)$, for all $x, y \in I$.

The following example demonstrates that R to be prime is essential in the hypotheses of the above results.

Example 3.4.1. Consider S as any ring. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\right\}$ and $I=\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S\right\}$ be a Lie ideal of R. Define F: $R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$. Then F is a generalized derivation with associated derivation d given by $d(x) = e_{11}x - xe_{11}$. It can be easily seen that R satisfies the properties: (i). $F(xy) - xy \in Z(R)$, (ii). $F(xy) + xy \in Z(R)$, (iii). $F(xy) - yx \in Z(R)$ and (iv). $F(xy) + yx \in Z(R)$ for all $x, y \in I$. However, I is not central.

Bergen et al. [40] proved that if R is a semiprime ring with unity and $d \neq 0$ is a derivation of R such that for every $x \in R$, $d(x)$ is zero or invertible in R, then R must be either a division ring D or $M_2(D)$, the ring of 2×2 matrices over a division ring D. Later, Bergen & Carini [38] extended this result to the case of Lie ideals. More precisely, they prove the following: Let R be a semiprime ring with unity, U a noncentral Lie ideal of R such that $D(U) \neq 0$, and $d(x)$ is either zero or invertible for every $x \in U$. Then R is either a division ring D or $M_2(D)$, for some division ring D. Since a noncentral Lie ideal of a simple ring R contains all the commutators $[x, y]$ with $x, y \in R$ except if R is of characteristic 2 and is 4-dimensional over its center, it is natural to check the case when $d(f(X_1 \cdots, X_k))$ is either zero or invertible for $X_i \in R$, where $f(X_1 \cdots, X_k)$ is a multilinear polynomial. Indeed, Lee [157] obtained the same conclusion as above by assuming that R is a semiprime ring and $f(X_1 \cdots, X_k)$ is not central-valued on R. On the other hand, Bergen [36] proved a result concerning a derivation with invertible or nilpotent values. It is shown that, if R is a ring without nonzero nil one-sided ideal, and d is a nonzero derivation such that $d(x)$ is invertible or nilpotent for all $x \in R$, then R is either a division ring or the ring of 2×2 matrices over a division ring. A full generalization in this vein was proven by Lee & Wong [161]. They showed that if d is a nonzero derivation and $f(X_1 \cdots, X_k)$ is a multilinear polynomial such that $d(f(X_1 \cdots, X_k))$ is either nilpotent or invertible for all X_i in some nonzero ideal of prime ring R, then R is either a division ring or the ring of 2×2 matrices over division ring, provided that R contains no nonzero nil one-sided ideals and $f(X_1 \cdots, X_k)$ is a multilinear polynomial not central-valued on R.

Recently, Komatsu & Nakajima [145] proved the following: Let R be a semiprime ring with unity and g be a generalized derivation of R. If $F(x)$ is zero or invertible for every $x \in R$, and $ker(F)$ contains no nonzero right ideals, then R must be either a division ring D or $M_2(D)$ for some division ring D. Very recently, Lin & Liu [163] extended the above mentioned result in the case generalized derivations. For more related results see for example ([10], [11], [53], [127], [138], [160], [186], [197], [206]).

3.5. Generalized Jordan Derivations

Let S be a non-empty subset of R. An additive mapping F: $R \longrightarrow R$ is said to be a *generalized Jordan derivation* on S if there exists a derivation d: $R \longrightarrow R$ such that $F(x^2) = F(x)x + xd(x)$, holds for all $x, y \in S$.

Clearly every generalized derivation on R is a generalized Jordan derivation. But the converse statement does not hold in general. It is shown in [13] that if R is a ring with a commutator which is not a divisor of zero, then every generalized Jordan derivation on a ring is a generalized derivation. In the year 2002, Ashraf et al. [16] obtained the conditions under which every generalized Jordan derivation on a ring is a generalized derivation. In fact, the result which we refer to states as follows:

Theorem 3.5.1. Let R be a 2-torsion-free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If F is an additive mapping of R into itself satisfying $F(u^2) = F(u)u + ud(u)$, for all $u \in U$, then $F(uv) = F(u)v + ud(v)$, for all $u, v \in U$.

Corollary 3.5.1. Let R be a 2-torsion-free prime ring and $F\colon R \longrightarrow R$ be a Jordan generalized derivation. Then F is a generalized derivation on R.

The following example due to Ashraf et al. [9] demonstrates that R to be prime is essential in the hypothesis of the above result.

Example 3.5.1. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Next, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$. Define a map $F\colon R \longrightarrow R$ such that $F \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Then with $d = 0$ and $U = R$, it can be easily seen that $F(r^2) = F(r)r = F(r)s = 0$ for all $r, s \in R$, but $F(rs) \neq 0$ for some $r, s \in R$.

The above theorem is still open for arbitrary Lie ideal.

In 2003, Jing & Lu ([134], Theorem 2.5) showed that every generalized Jordan derivation on a 2-torsion-free prime ring is a generalized derivation. An additive mapping $f\colon R \longrightarrow R$ is said to be a *generalized Jordan triple derivation* if there exists a Jordan triple derivation $\delta\colon R \longrightarrow R$ satisfying $f(aba) = f(a)ba + a\delta(b)a + ab\delta(a)$ for all $a, b \in R$. Further, they obtained some more general results:

Theorem 3.5.2. Let R be a 2-torsion-free prime ring, then every generalized Jordan triple derivation on R is a generalized derivation.

Theorem 3.5.3. Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices and B be an arbitrary algebra over the complex field $\mathbb{C}$. Suppose that $\delta\colon M_n(\mathbb{C}) \longrightarrow B$ is a linear mapping such that $\delta(P) = \delta(P)P + P\tau(P)$ holds for all idempotent P in $M_n(\mathbb{C})$, where $\tau\colon M_n(\mathbb{C}) \longrightarrow B$ is a linear mapping satisfying $\tau(P) = \tau(P)P + P\tau(P)$, for any idempotent P in $M_n(\mathbb{C})$, then δ is a generalized Jordan derivation. Moreover, δ is a generalized derivation.

In same paper [134], Jing & Lu proved some more related results and posed two questions which are open problems till date.

- **Open Problem 3.5.1**: If R is a 2-torsion-free semiprime ring, then every generalized Jordan derivation on R is a generalized derivation R.
- **Open Problem 3.5.2**: If R is a 2-torsion-free semiprime ring, then every generalized Jordan triple derivation on R is a generalized derivation.

Inspired by the definition of (θ, ϕ)-derivation, the notion of generalized (θ, ϕ)-derivation was extended by Ashraf et al. [9] as follows:

Definition 3.5.1. Let S be a non-empty subset of R. An additive mapping $F\colon R \longrightarrow R$ is called a *generalized (θ, ϕ)-derivation* (resp. *generalized Jordan (θ, ϕ)-derivation*) on S if there exists a (θ, ϕ)-derivation d: $R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ holds for all $x, y \in S$ (resp. $F(x^2) = F(x)\theta(x) + \phi(x)d(x)$ holds for all $x \in S$).

Remark 3.5.1. Every generalized $(1, 1)$-derivation (resp. generalized Jordan $(1, 1)$-derivation) on R is generalized derivation (resp. generalized Jordan derivation) on R, where 1 is the identity mapping on R.

Clearly, every generalized derivation on a ring R is a generalized Jordan derivation on R. But the converse of this statement need not be true in general. The following example due Ali [2] justifies this fact:

Example 3.5.1. Let R be a noncommutative ring and $a, b \in R$ such that $xax = 0$ and $x^2a = 0$, for all $x \in R$ but $xay \neq 0$, for some x and y, $(x \neq y) \in R$. Define maps F: $R \longrightarrow R$ as follows: $F(x) = xa + bx$. Then there exists an inner derivation d_a: $R \longrightarrow R$ such that $d_a = [a, x]$. It is readily verified that F is a generalized Jordan derivation but not a generalized derivation.

Very recently, in the year 2004, Ashraf et al. [9] proved the following results on Lie ideals:

Theorem 3.5.3. Let R be a 2-torsion-free prime ring and U a noncommutative Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one, onto and d is a (θ, ϕ)-derivation of R. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Theorem 3.5.4. Let R be a 2-torsion-free prime ring and U a nonzero commutator Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that θ is an automorphism of R and d is a (θ, θ)-derivation. If F: $R \longrightarrow R$ is a generalized Jordan (θ, θ)-derivation on U, then F is a generalized (θ, θ)-derivation on U.

As a consequence of above, we have

Corollary 3.5.2. Let R be a 2-torsion-free prime ring and F: $R \longrightarrow R$ a generalized Jordan derivation on R. Then F is a generalized derivation on R.

If the underlying ring R is arbitrary, then the following result was obtained in [9]:

Theorem 3.5.5. Let U be a Lie ideal of a 2-torsion-free ring such that $u^2 \in U$, for all $u \in U$. Suppose that θ, ϕ are endomorphisms of such that θ is one-one and onto. Suppose further that U has a commutator which is not a zero-divisor. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Corollary 3.5.3. Let R be a 2-torsion-free ring and let F: $R \longrightarrow R$ a generalized Jordan derivation on R. If R has a commutator which is not a zero-divisor, then F is a generalized derivation on R.

Remark 3.5.2. Since every ideal in a ring R is a Lie ideal of R, the conclusion of the above theorems hold when U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, there exist Lie ideals with this property which are not ideals. For example, consider an any ring R and U is the additive subgroup of R generated by the idempotents of R. If e is an idempotent in R, and $x \in R$ then it is easy to see that, $u = e + ex - exe$ and $v = e + xe - exe$ are idempotents. Hence, $ex - xe = u - v \in U$. Thus, U is a Lie ideal of R.

To conclude this section, let us mention few problems concerning such possible extensions of the above theorems:

Open Problem 3.5.3. Let R be a 2-torsion-free prime ring and U a Lie ideal of R. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one and onto. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Open Problem 3.5.4. Let R be a 2-torsion-free semiprime ring and U a Lie ideal of R. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one and onto. If F: $R \longrightarrow R$ is a generalized Jordan triple (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

A more precise description of the generalized derivations and Jordan generalized derivations take up a lot of space, so we feel that it is better to resist the temptation to express this subject in greater details and instead refer to [13], [16], [127], and to some of the most recent articles [9], [134], [145], [183], [184] for the advanced theory.

3.6. Lie Derivations and Lie Rings

For almost 30 years, the study of Lie isomorphisms and Lie derivations was carried on mainly by Martindale III and his students. In 1964 Martindale, generalizing an unpublished result of Kaplansky (obtained in the case of a matrix ring over a field), described Lie derivations of primitive rings of characteristic not 2 with nontrivial idempotents [168]. In subsequent papers of several authors, the analogous problem was considered either in the context of prime rings with involution [212] or in the context of von Neumann algebras under a similar assumption.

In the year 1961, Herstein [112] in his AMS hour Talk, titled "Lie and Jordan Structure in Simple, Associative Rings", posed a number of problems on Lie (Jordan) isomorphisms and derivations. In [28], Beidar & Chebotar considered two of them:

- **Problem 3.6.1**: Describe the Lie derivations of prime rings ([112], Problem 3);
- **Problem 3.6.2**: Given a prime ring A, describe the Lie derivations of $[A, A]$ and $[A, A]/Z([A, A])$, where $Z(R)$ denotes the center of a ring R ([112], Problem 4).

In 1993, Brešar [48] solved Problem 3.6.1 under the assumption that the prime ring in question does not satisfy St_4, the standard polynomial identity of degree 4. It was the first time that functional identities[1] were applied to obtain the description of Lie isomorphisms and Lie derivations. Since then the method of functional identities has been further developed (see [27] for a historical account) and has been successfully applied to such problems in several papers. In 1997, Banning & Mathieu [23] extended to semiprime rings the description of Lie derivations obtained by Brešar in the prime case.

Beidar & Chebotar [28] considered F a commutative ring with 1, A a prime F-algebra with Martindale extended centroid C and with central closure A_C and R a noncentral Lie ideal of the algebra A generating A. Further, they considered $\overline{R} = R/Z(R)$ the factor Lie algebra and $\delta\colon \overline{R} \to \overline{R}$ a Lie derivation, supposing that $char(A) \neq 2$ and that A does not satisfy St_{14}, the standard identity of degree 14. They showed that $R \cap C = Z(R)$ and that there exists a derivation of algebras $D\colon A \to A_C$ such that $x^D + C = (x + C)^\delta \in (R + C)/C = \overline{R}$ for all $x \in R$. This result solves Problem 3.6.2.

Roughly speaking, the above mentioned descriptions say that (if some requirements are satisfied) a Lie derivation of a ring R has the form $\delta + \tau$, where δ is an ordinary derivation from R to an enlargement R' of R and τ is an additive map from R to the center of R'. Unfortunately R' may be too large. In fact, the enlargement R' is usually too large to be useful in the study of the analytical properties of Lie derivations on general Banach algebras.

In [220], Villena considered D a Lie derivation on an unital complex Banach algebra. Then for every primitive ideal P of A, except for a finite set of them which have finite codimension greater than one, there exists a derivation d from A/P to itself and a linear functional τ on A such that $Q_P D(a) = d(a+P)+\tau(a)$, for all $a \in A$ (where Q_P denotes the quotient map from A onto A/P). Moreover, the preceding decomposition holds for all primitive ideals in the case where D is continuous. It is important to note that the properties of ordinary derivations on primitive Banach algebras will be (almost) inherited (modulo the center and the radical) by Lie derivations on unital complex Banach algebras.

On the other hand, Jordan & Jordan [135] studied how the ideal structure of the Lie ring of derivations of an associative ring R, denoted by $D(R)$, is determined by the ideal structure of R. If R is a simple (resp. semisimple) finite-dimensional $Z(R)$-algebra and $\delta(z) = 0$ for all $\delta \in D(R)$, then every derivation of R is inner and $D(R)$ is known to be a simple (resp. semisimple) Lie algebra (see [122], [131]). Jordan's interest was centred in extending these results to the case where R is a prime or semiprime ring.

Prime and semiprime ideals of Lie rings have been studied by Brown & McCoy, in 1958, and by Kawamoto, in 1974. Let now R be a commutative ring with identity and δ be a derivation of R. Then the set, $R\delta$, of all derivations of R of the form $r\delta\colon x \to r\delta(x)$, $r \in R$, is a Lie subring of the Lie ring $D(R)$ of derivations of R. In [135], Jordan & Jordan studied the structure of $D(R)$ and found that $R\delta$ played an analogous role to that played by Lie ring $I(S)$ of inner derivations of a noncommutative ring S in the study of $D(S)$. Furthermore, it was shown in [135] that the properties of $R\delta$ closely resemble the known properties of $I(S)$. In particular,

[1]For more details on functional identities, see ([27], [30], [54]).

it was shown that the following results hold in the case where R is 2-torsion-free: (i). If R is prime or If R is δ-prime noetherian, then $R\delta$ is a prime Lie ring; (ii). If R is δ-simple noetherian, then $R\delta$ is a simple Lie ring. In [136], the authors continued the study of the structure of the Lie ring $R\delta$ and of certain of its Lie subrings.

3.7. Nil, Nilpotent, and Composition of Derivations

The notion of nil derivations is a generalization of the notion of nilpotent derivations. The latter, because of its close relation with automorphisms and the existence of a Jordan decomposition into semisimple and nilpotent parts for a large family of derivations (it is a generalization of that of algebraic derivations), has received considerable attention recently (see [71]). Based on Chung [71], for a prime ring of characterisitc zero, a relation between a nil derivation being inner with the existence of nontrivial fixed points of its corresponding automorphism was established. From this, the criterion on ∂ being "inner" and induced by a nil element was derived. As an application, the result that a nilpotent derivation is induced by a nilpotent element in the endomorphism ring $End(I_R, I_R)$, where I_R is certain ideal of R was deduced. This is a generalization of some well-known results due to Kharchenko and others. This problem was not yet studied for higher derivations.

Now, let us consider R be a ring and d a derivation of R. We say that d is *locally nilpotent* if for any $\alpha \in R$ there exists $n \in I\!N$ such that $d^n(\alpha) = 0$. Following Ferrero, Lequain & Nowicki [93], locally nilpotent derivations play an important role in commutative algebra and algebraic geometry, and several problems may be formulated using locally nilpotent derivations. In particular, they play an important role in the Jacobian conjecture. It is well-known (by the works of Nousiainen, Nowicki, Sweedler, and Wright) that the Jacobian problem is equivalent to the problem of local nilpotence of some $\mathbb{C}$-derivations in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The problem is still open even for the 2-variable case $\mathbb{C}[x, y]$. If $d = \frac{\partial}{\partial x}$ and $\delta = \frac{\partial}{\partial y}$ are the partial derivatives in $\mathbb{C}[x, y]$, then every $\mathbb{C}$-derivation Δ of $\mathbb{C}[x, y]$ has the form $\Delta = ad + b\delta$, where $a, b \in \mathbb{C}[x, y]$ are uniquely determined. The derivations d and δ are locally nilpotent and they commute. It now appears to be of interest to get necessary and sufficient conditions on a and b for Δ to be locally nilpotent. In [93] the authors found them for a commutative, reduced, $\int$-torsion-free ring R with an identity element and where d and δ are two locally nilpotent derivations which commute, and for $b \in R$ such that $\delta(b) = d(b) = 0$. They gave a partial answer that includes the cases $b = 0$ and $b = 1$. The condition is that $d(a) = 0$, where Δ is a derivation $ad + b\delta$ with $a \in R$.

In the same year, 1992, Lanski [155] combined the 1978's results and ideas of Kharchenko [143], showing that certain algebraic derivations of prime rings are inner, with those of 1983 of Martindale & Miers [170] which showed that nilpotent inner derivations are obtained from nilpotent elements of index of nilpotence roughly half that the index of the derivation. More specifically, Lanski considered a derivation d which is nilpotent on certain subsets of a prime ring R: namely, on Lie ideals, right ideals and when R has an involution, on the set of symmetric or skew-symmetric elements of R. Using an earlier own work [150], he showed that d must be inner in the Martindale quotient ring of R, and then using the ideas in [152] to see that d can be given by a nilpotent element whose index of nilpotence depends on that of d, the subset in question, and the characteristic of R.

On the other hand, in [110] Herstein proved that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n a fixed integer, then $d = 0$. As we wrote earlier, in [97] Giambruno & Herstein extended this result to arbitrary derivations in semiprime rings. In [65] Carini & Giambruno proved that if R is a prime ring with a derivation d such that $d(x)^{n(x)} = 0$ for all $x \in U$, a Lie ideal of R, then $d(U) = 0$ when R has no nonzero nil right ideals, $char(R) \neq 2$ and the same conclusion holds when $n(x) = n$ fixed and R is a 2-torsion-free semiprime ring. Using the ideals in [65] and the methods in [89], Lanski [152] removed both the bound on the indices of nilpotence and the characteristic assumptions on R.

In [46], Brešar gave a generalization of the result due to Giambruno & Herstein [97] in another direction. Explicitly, he proved the theorem: Let R be a semiprime ring with a derivation d, $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$ when R is an $(n-1)!$-torsion-free ring. Lee & Lin [159] were motivated by Brešar's result and by Lanski's paper [155]. They proved Brešar's result without the assumption of $(n-1)!$-torsion-free on R. In fact, they studied the Lie ideal case given in [155] and then obtained Brešar's result as the corollary to their main theorem. A good account of this subject could be found in [149].

In the year 2002, Chuang & Lee [69] considered a prime GPI-ring R with extended centroid C. They proved that if C is a finite field, then there exist nonzero derivations $\delta_1, \ldots, \delta_n$ of R satisfying $\delta_1(x)\delta_2(x)\ldots\delta_n(x) = 0$ for all $x \in R$. This answer a problem posed by Brešar, Chebotar & Šemrl [55]. Moreover, the authors generalized their theorem to the case of generalized derivations with assumption on Lie ideals.

4. Some Applications

The theory of derivations and automorphisms of an associative rings is a direct descendant of the development of classical Galois theory (cf. Suzuki [211], Taelman [216] and Van der Put & Singer [217] for details) and the theory of invariants. The theory of derivations and automorphisms plays an important role not only in ring theory, but also in functional analysis; linear differential equations, concerning the question of innerness and outerness, for instance, the classical Noether-Skolem theorem yields the solution of the problem for finite dimensional central simple algebras (see [113]). An extensive and deep theory has been developed especially for derivations of C^*-algebras, commutative Banach algebras and Galois theory of linear differential equations (see; e.g., Bonsall & Duncan [43], Murphy [182] - a more recent condensed survey, Frank [96], Pedersen [195] and Sakai [199]). Especially in analysis it is customary to treat derivations of one algebra into a bigger one (into a bimodule). To explain more precisely, we have the following:

4.1. Some Nowicki's Results Concerning Derivations Closely Connected with the Ring of Constants

Nowicki [191] works with k-derivations of the polynomial ring $k[X] = k[x_1, \ldots, x_n]$ over a field k of characteristic zero. The object of his principle interest is $k[X]^d$, the ring of constants of a k-derivation d of $k[X]$, that is, $k[X]^d = \{f \in k[X];\ d(f) = 0\}$.
Assume that $f_1, \ldots, f_n$ are polynomials belonging to $k[X]$. There exists then a unique k-derivation d of $k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. The derivation d is defined by

$$d(h) = f_1\frac{\partial h}{\partial x_1} + \ldots + f_n\frac{\partial h}{\partial x_n}, \text{ for } h \in k[X]. \tag{1}$$

Now, consider a system of polynomial ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \ldots, x_n(t)),\ 1 \leq i \leq n. \tag{2}$$

If k is a subfield of the complex numbers $\mathbb{C}$, then it is evident what the system means. When k is arbitrary then it also has a sense. This system has a solution in $k[[t]]$, the ring of formal power series over k in the variable t (see ([191], Section 1.6)).
Let $k(X) = k(x_1, \ldots, x_n)$ be the quotient field of $k[X]$. An element h of $k[X] \setminus k$ (resp. of $k(X) \setminus k$) is said to be a *polynomial* (resp. *rational first integral* of the system (2)) if the following identity holds

$$f_1\frac{\partial h}{\partial x_1} + \ldots + f_n\frac{\partial h}{\partial x_n} = 0. \tag{3}$$

Thus, the set of all the polynomial first integrals of (3) coincides with the set $k[X]^d \setminus k$ where d is the k-derivation defined by (1). Moreover, the set of all the rational first integrals of (2) coincides with the set $k(X)^d \setminus k$, where $k(X)^d = \{h \in k(X); d(h) = 0\}$ and where d is the unique extension of the k-derivation (2) to $k(X)$.

In various areas of applied mathematics (as well as in the theoretical physics and chemistry) there occur autonomous systems of ordinary differential equations of the form (3). There arises the following question: "do there exist first integrals of a certain type, for example, polynomial or rational first integrals?" This problem has been studied intensively for a long time; see for example ([122], [193], [202], [210]) where many references on this subject can be found. The problem is known to be difficult even for $n = 2$.

Computers are frequently used in solving this problem. There are computer programs which make it possible to find all the polynomial first integrals up to a given highest degree r but they do not provide any information beyond r.

In this section, we use the vocabulary of differential algebra ([139], [146]). In terms of derivations the above problem consists in the finding of methods leading to the statment whether the ring of the form $k[X]^d$ (or $k(X)^d$), where d is a given k-derivation of $k[X]$, is nontrivial i.e., different than k). A certain result containing some necessary and sufficient conditions (even for $n = 2$) on polynomials $f_1, \dots, f_n$ would be desirable and remarkable for the derivation defined by the formula (1) to possess a nontrivial ring of constants.

There exist other natural problems concerning the discussed question. Assume that d is a k-derivation of $k[X]$ such that $k[X]^d \neq k$. Then there arises the following question: Is the ring $k[X]^d$ finitely generated over k? This question is a special case of the fourteenth problem of Hilbert ([126], [188]). Let us stress that there exist k-derivations of $k[X]$ for which the ring of constants is not finitely generated (see [191], Section 4.2). How to decide whether a given k-derivation of $k[X]$ has a finitely generated ring of constants?

Suppose that we already have one such derivation which has a finitely generated ring of constants. How can one find its finite (possibly smallest) generating set? Can the minimal number of generators be limited in advance? What can be said about this number?

Evidently, not every k-subalgebra of $k[X]$ is a ring of constants with respect to a certain k-derivation (or a family of k-derivations) of $k[X]$. For example, $k[x_1^2, \dots, x_n^2]$ is a such subalgebra. Therefore, a question arises which subalgebras are the rings of constants. Does there exist an algebraic description of such subalgebras? Let D be a family of k-derivations of $k[X]$. Consider the ring of constants

$$k[X]^D = \bigcap_{d \in D} k[X]^d = \{w \in k[X]; d(w) = 0, \text{for all } d \in D\}.$$

Does there exist a k-derivation δ of $k[X]$ such that $k[X]^D = k[X]^\delta$? Similar questions can be asked for all the subfields of the field $k[X]$. All the above questions will constitute a group dealt with in [191]. A. Nowicki also presented other issues related to the constant rings in $k[X]$. In particular, we presented:

- methods leadings to the proof that some polynomial derivations do not possess a nontrivial polynomial (often even rational) constant as well as methods for the finding of a finite set of generators, illustrated by numerous examples;
- an algebraic description of all the subrings of $k[X]$ which are rings of constants of derivations. Moreover, applications of the description to the above mentioned problems of the finiteness and the minimal number of generators.

Later, in 2004, Ollagnier & Nowicki [193] considered the following problem: Let d_1: $k[X] \to k[X]$ and d_2: $k[Y] \to k[Y]$ be k-derivations, where $k[X] = k[x_1, \dots, x_n]$ and $k[Y] = k]y_1, \dots, y_m]$ are polynomial algebras over a field k of characteristic zero. Denoting by $d_1 \oplus d_2$ the unique k-derivation of $k[X, Y]$ such that $d \mid_{k[X]} = d_1$ and $d \mid_{k[Y]} = d_2$, they proved that if d_1 and d_2 are *positively homogeneous* and if d_1 has no nontrivial Darboux polynomials, then every Darboux polynomial of $d_1 \oplus d_2$ belongs to $k[Y]$ and is a Darboux polynomial of d_2. Moreover, the authors proved a similar fact for the algebra of constants of $d_1 \oplus d_2$ and presented several applications of their results.

4.2. Derivations in Skew Polynomial Rings

Let R be a commutative ring with identity and d a derivation of R. Consider the set S of all polynomials on one variable, say x, over R and define in S addition in the usual way and multiplication by the rule $xr = rx + d(r)$ for all $r \in R$. Then it is well-known that S becomes a ring denoted by $R[x, d]$, and it is called a *skew polynomial ring* (cf. Cohn [74] for details). For derivations $d_1, d_2, d_3, ..., d_n$ of R, one can also construct a skew polynomial ring in n variables of R, $R_n = R[x_1, x_2, ..., x_n; d_1, d_2, d_3, ..., d_n]$ such that $x_i r = r x_i + d_i(r)$ and $x_i x_j = x_j x_i$ for any $r \in R$. The properties of these skew polynomial rings have been discussed by many authors (see for example; Cozzen [78], Hamanchi & Nakajima [105], Jordan [137] and

Voskoglou [221], [222]. In [221], Voskoglou has given the properties of the skew polynomial ring over a ring R of prime characteristic which are connected with the D-simplicity of R with respect to a set of derivations D of R.

Let k be a field of characteristic zero, $F = k((Y))$ the local field of Laurent series in one indeterminate Y, and ∂_Y the usual derivation of F. In 1992, Dumas & Vidal [86] described completely the k-derivations of $K = F((X, \partial_Y))$. As an application, they studied the structure of the higher derivations in skew rings of characterisitc zero. This subject could be depened, since that Dumas & Vidal constructed a new ring $K[[X, D]]$, called the Cohen ring, following Vidal [219].

Deformations of a polynomial algebra, such as the Weyl algebra or functions on quantum affine space, may be expressed by formulas involving derivations of the polynomial algebra. These formulae are power series in an indeterminate with coefficients in the universal enveloping algebra of the Lie algebra of derivations. There are generalizations of such deformations to other types of algebras, such as functions of a manifold or orbifold, that are of current interest. In [228], Witherspoon gave a new generalization of the formulas themselves and applied them to crossed products of polynomial algebras with groups of linear automorphisms. These group crossed products are of interest in geometry due to their relationship with corresponding orbifolds. Particular deformations of such crossed products, called graded Hecke algebras (firstly defined by Drinfel), have been studied by many authors, for example for crossed products with real reflection groups. For these crossed product algebras, the universal enveloping algebra of the Lie algebra of derivations does not capture all the known deformations. Instead, she derives a deformation formula from the action of a bialgebra or Hopf algebra under some hypotheses, recovering more of these known deformations as well as some new ones.

In the year 2003, Taelman [216] observe that the Dieudonne determinant induces a non-negative degree function on the ring of matrices over a skew polynomial ring. Then, he apply this degree function to calculate the dimension of the solution space of linear matrix differential equations in the following way: Let F be a differential field of characteristic 0. This means F is equipped with an additive map (called derivation i.e., $d(ab) = d(a)b + ad(b)$, for all $a, b \in F$). Let $C \subset F$ be a field of constant, that is the kernel of derivation. Assume that the derivation is nontrivial i.e., $C \neq F$. Examples are $F = C(x)$ and $F = C((x))$ with the usual derivation. Now, we consider the skew polynomial ring $R = F[\partial, 1']$ with center C. It acts F-linear on differential field extension of F by $\partial(a) = d(a)$. A homogeneous matrix differential equation of the form

$$A_0 y + A_1 y' + A_2 y'' + \cdots + A_d y^d = 0$$

where y denotes a vector in F^n and the A_i are matrices in $M(n, F)$ can be written as $Ay = 0$ with $A = \sigma A_i \partial^i \in M(n, F)$. Conversely, every A corresponds to such a differential equation. As in the proof of Theorem 1.1 of [216], we associate with A the R-module $M := R^n / R^n A = R^{(s)} \oplus M_{tors}$. Take $F \subset l$ to be the Picard-Vessiot extension of M_{tors} or alternatively, take l to be a universal differential field extension of F. When $s = 0$, all solutions of the differential equation exist over l. The contravariant solution space V of M is defined to be the C-vector space $V := Hom_R(M, l)$. It is finite-dimentional if and only if $s = 0$, and in that case it is dual to the C-space of solutions in l^n of the given differential equation. Therefore, he obtained the following relationship:

$$dim_C V = dim_F M = deg\ det A.$$

Finally, he remarked that the completely analogous results hold for difference and q-difference equations.

In 2005, Cortes [76] studied generalizations of McCoy's theorem in skew polynomial rings. He obtained that the bijection between the set of right annihilator in a ring R and the set of a right annihilator in $R[x; \sigma]$, where σ is an automorphism of R, is equivalent to R be skew Armendariz ring. Moreover, Cortes studied the relationship between Baerness, right Goldie property and right p.p.-property of R and $R[x; \sigma]$ using the concept of a skew Armendariz ring. Further, he studied the properties of quasi-skew Armendariz rings.

4.3. Algebraic, Integral, and p-Integral Derivations

Following Lanski [150], the first general result on algebraic derivations was obtained in 1957 by Amitsur [3], who proved that an algebraic derivation of a simple ring of characteristic zero must be inner. An extension of this result to prime rings was proved in 1978 by Kharchenko [143] in a celebrated paper, using his work on differential identities of prime rings.

Later on 1985, Lanski [153] extended the work of Kharchenko and studied differential identities of ideals in prime rings, and of the set of (skew) symmetric elements in ideals of prime rings with involution. As a result of this work, he showed that a derivation of a prime ring R is algebraic if its restriction to an ideal of R is algebraic, and also must be inner if the characteristic of R is zero. Furthermore, when R has an involution, then any derivation of R, algebraic when restricted to the (skew) symmetric elements of an ideal of R, must be inner when the characteristic of R is zero, and algebraic if the characteristic of R is positive. In this last result, the question of whether the derivation must be algebraic in characteristic zero was unanswered. He proved that it must be algebraic. In his paper [150], Lanski considered derivations which are algebraic when considered as endomorphisms of certain subsets of prime rings. He proved the results using the theory developed in [153] to extend and strengthen to Lie ideals the results there on ideals. Specifically, he showed that if a derivation of a prime ring R satisfies a polynomial when restricted to a noncommutative Lie ideal of R, then the derivation satisfies the same polynomial, as an endomorphism of R. This generalizes, to Lie ideals, a result of Chung & Luh [72] on nilpotent derivations on ideals of R.

There is an interesting problem the study of the algebraic derivations d defined in a prime ring R (with unity), and they respectively extensions $d^\star$ to the left Martindale quotient ring of R, denoted by Q. There are several papers in this line, from where we choose ([86], [142], [143], [147], [164]).

It is well-known that if R is a semiprime ring and d: $R \to R$ is a derivation, then d can be uniquely extended to a derivation $d^\star$: $Q \to Q$ (i.e., such that $d^\star\,|_R = d$). When R is a prime ring, Leroy & Matczuk [164] in 1985, and later in 1992 by Ouarit on when R is semiprime [194], different types of algebraicity were related. It was proved that the following conditions are equivalent: d is R-algebraic; $d^\star$ is R-algebraic; d is Q-algebraic; $d^\star$ is Q-algebraic; d is C-algebraic (where C indicates the extended centroid of R); $d^\star$ is C-algebraic. Also it is a well-known 1978's Kharchenko result [143] that if d is algebraic over a prime ring R, then d is X-inner, provided that $char(R) = 0$ or $char(R) = p$ is greater than the degree of algebraicity of d. Further, Ferrero & Haetinger ([91], [104]) extended this result to higher derivations.

Later, in 1987, Nowicki [190] published a work on integral derivations: If p is a prime number and $k \subset R$ are commutative rings of characteristic p, then we say that a k-derivation d of R is *p-integral* over a d-subring A of R if there exists a finite set $\{a_0, \ldots, a_{n-1}\}$ of elements of A such that $d^{p^n} + a_{n-1}d^{p^{n-1}} + \ldots + a_1 d^p + a_0 d = 0$. The author described the p-integral k-derivations of R by the Lie p-subalgebras of $Der_k(R)$ $(d(k) = 0)$. He proved in ([190], Proposition 2.2) that if R is noetherian and $Der_k(R)$ is finitely generated as an R-module then every k-derivation of R is p-integral over R. In particular, if k is noetherian and R is either the ring $k[x_1, \ldots, x_n]$ of polynomials or the ring $k[[x_1, \ldots, x_n]]$ of formal power series over k, then every k-derivation of R is p-integral over R. The main result of the paper ([190], Theorem 4.1) shows that, in the cases of polynomials or power series, every such k-derivation d is p-integral over the ring of constants, even if k is non-noetherian, and that the minimal polynomial for d is of the degree p^m, where $m \leq n$ (the number of variables).

In 1991, Ferrero & Nowicki [93] studied locally integral derivations and endomorphims of commutative rings. A derivation d of a ring R is said to be *locally integral* if, for every $a \in R$, there exists $m = m(a) \in I\!N$ such that $d^m(a)$ is contained in the ideal of R generated by a, $d(a), \ldots,$ $d^{m-1}(a)$. A locally integral endomorphism of R is defined similarly. The authors presented conditions for a derivation to be locally integral, as well as they included several examples of K-derivations and K-endomorphisms of finitely generated algebras and power series rings, where K is a commutative ring with an identity and R is a commutative K-algebra.

It is well-known, by results of Berman and Sweedler, that the actions both of finite groups and finite dimensional Lie algebras on algebras have common generalizations. Namely, these are examples of actions of Hopf algebras. Thus we can look at connections between behavior of automorphisms and derivations via Hopf algebras. On the other hand, if we have a finite dimensional real or complex algebra A, then the group of all automorphisms of A forms a Lie group and its Lie algebra is equal to the Lie algebra of all derivations of A. Thus, there is a very nice correspondence between derivations and automorphisms of A given by exponential and logarithm maps. Still in 1987, Matczuk [171] presented an analog of these maps for algebraic derivations and automorphisms. He also gave some applications of this construction to investigation of algebraic automorphisms of prime algebras.

4.4. Derivations on Lie Ideals

In the early 1950's, Herstein initiated a study of the Jordan and Lie ideals of R in case that R was a simple associative ring (either without or with an involution). In the ensuring years his work has been generalized in various directions, on the one hand, to the setting of prime and semiprime rings, and, on the other hand, to invariance conditions other than given by ideals.

In 1986, Martindale & Miers [169] wrote the Herstein's Lie Theory revisited. Part of their motivation was to obtain the Lie ideal theory for semiprime rings with involution by a somewhat different approach from the self contained, elementary, very clever methods embodied in the original style of Herstein. And they did it.

Beidar, Brešar & Fong ([24], [26]), in 2001, continuated a project initiated by Brešar & Šemrl [57] in 1999, where the main idea was to connect the concept of dense action on modules with the concept of outerness of derivations and automorphisms. In particular, one can view their results as generalizations of the Chevalley-Jacobson density theorem. This celebrated theorem is one of the important tools of rings theory and has already been generalizated in various directions, as the reader can see in [24]. In [26], the authors considered a Lie ideal of a ring acting on simple modules via multiplication. Their goal were to extend to this context results obtained in [24]. They confined theirselves with the case of automorphisms. As an application they generalized results of Drazin on primitive rings with pivotal monomial to primitive rings whose noncentral Lie ideal has a pivotal monomial with automorphisms. The authors noted that while Martindale's results on prime rings with generalized polynomial identity were extended to prime rings with generalized polynomial identities involving derivations and automorphisms, the corresponding program for results of Amitsur and Drazin on primitive rings with (generalized) pivotal monomials has not been done. In this paper they made the first step in this direction.

In 1981, Bergen, Herstein & Kerr [40] considered the relationship between the derivations and Lie ideals of a prime ring. They also looked at the action of derivations on Lie ideals; the results they obtained extended some that had been proved earlier only for the action of derivations on the ring itself. Let R be a ring and $d \neq 0$ a derivation of R. If U is a Lie ideal of R, they were concerned about the size of $d(U)$. How does one measure this size? One way is to look at the centralizers of $d(U)$ in R; the bigger $d(U)$, the smaller this centralizer should be. This explains the interest in the centralizers of $d(U)$. The result obtained in [40] generalized the principle theorem of [109]. They also measured the size of $d(U)$ by looking at how large $\overline{d(U)}$, the subring generated by $d(U)$, turns out to be, generalizing results of [108]. Furthermore, a well-known and often used result states that if d is a derivation of R, which is semiprime and 2-torsion-free, such that $d^2 = 0$ then $d = 0$. If R is prime, $char(R) \neq 2$, and $d^2(I) = 0$ for a nonzero ideal I of R, it also follows that $d = 0$. What can be say if $d^2(U) = 0$ for some noncentral Lie ideal of R? For inner derivations this was studied and answered in [114]. For prime rings and for any derivation $d \neq 0$ Bergen, Herstein & Kerr answered the question of when $d^2(U) = 0$ completely in Theorem 1 of [40].

Usually, the theory of prime rings operates with Lie ideals that do not lie at the center of the ring. For an effective operation with semiprime rings, a stronger concept is desirable. Specifically, we will say that a Lie ideal is *essentially noncentral* if its intersection with any nonzero associate ideal does not lie at the center of the ring. Note that if R is a prime ring and $2R \neq 0$, then R is a ring without 2-torsion. On the other hand, if R is a prime ring, $2R \neq 0$ and U is a its Lie ideal not lying at the center, then U is essentially noncentral. In [17], Avraamova considered R a semiprime ring without 2-torsion and U its essentially noncentral Lie ideal. If on U the polynomial identity of degree n is satisfied, then he proved that R satisfies identity of degree $2n$. Furthermore, he extended to semiprime rings the first and the second Posner's theorems.

4.5. Derivations Having Values Satisfying Certain Properties

There are another line of investigating in the literature concerning derivations having values satisfying certain properties.

Bergen, Herstein & Lanski [40], in 1983, studied a question which, although somewhat special, has the virtue that its answer can be given in a very precise, definitive, and succinct way. They showed that the structure of a ring is very tightly determined by the imposition of a special behavior on one of its derivations. That is, they classified the semiprime ring R possessing a nonzero derivation d such that $d(x)$ is either 0 or invertible for all $x \in R$. They proved that R is either a division ring or the ring of 2×2 matrices over a division ring. Later, in 1988, Bergen and Carini [39] obtained the same conclusion assuming that $d(x)$ is 0 or invertible merely for all x in some noncentral Lie ideal of R. In 1993, Lee [157] extended this result by studying the more general situation when $d(f(x_1, \ldots, x_t))$ is either 0 or invertible for all $x_1, \ldots, x_t$ in R, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial not central-valued on R.

As to derivations having nilpotent values, Felzenszwalb and Lanski [89] proved that, if R is a prime ring with no nonzero nil one-sided ideals and d is a derivation such that $d(x)$ is nilpotent for all x in some nonzero ideal of R, then $d = 0$. The extensions of this theorem to Lie ideals were obtained by Carini and Giambruno [65] in the case of $char(R) \neq 2$, and by Lanski [152] in the case of arbitrary characteristic. In the year 1996, Wong [226] proved a full generalization of this result. In fact, in [226] it is shown that if $d(f(x_1, \ldots, x_t))$ is nilpotent for all $x_1, \ldots, x_t$ in some nonzero ideal of R, where $f(X_1, \ldots, X_t)$ is a multinilear polynomial not central-valued on R, then $d = 0$.

On the other hand, Bergen [37] proved a result concerning a derivation with invertible or nilpotent values. It was shown that, if R is a ring with no nonzero nil one-sided ideals and d is a nonzero derivation on R such that $d(x)$ is invertible or nilpotent for all x in R, then R is a division ring or the ring of 2×2 matrices over a division ring. In 2000, Lee and Wong [161], the authors considered the situation when $d(f(x_1, \ldots, x_t))$ is invertible or nilpotent for all $x_1, \ldots, x_t$ in some nonzero ideal of a prime ring, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial no central-valued on R.

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