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HIGHER DERIVATIONS OF SEMIPRIME RINGS

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ABSTRACT

In this paper we study higher derivations of prime and semiprime rings satisfying linear relations. We extend several results which are known for algebraic derivations and we prove some other results.

INTRODUCTION

Derivations of prime and semiprime rings have been extensively studied in the last 50 years. In particular, algebraic derivations were considered in several papers ([6–8, 13]).

Let R be a semiprime ring, Q the left Martindale's quotient ring of R and C the extended centroid of R. If $d : R \to R$ is a derivation of R, then d can be uniquely extended to a derivation $d^* : Q \to Q$. When R is a prime ring [8] and later on when R is semiprime [13], different types of algebraicity were related. It was proved that the following conditions are equivalent: d is R-algebraic; $d^\star$ is R-algebraic; d is Q-algebraic; $d^\star$ is Q-algebraic; d is C-algebraic; $d^\star$ is C-algebraic. Also it is a well-known result that if d is algebraic over a prime ring R, then d is X-inner [6], provided that $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) = p$ is greater than the degree of algebraicity of d.

On the other hand, higher derivations (HD, for short) of a ring R have been studied in many papers mainly in commutative rings (see, for example, [1, 9–11]), but also in non-commutative rings ([2] and [3]). Note that if $d : R \to R$ is a derivation of an algebra R over the field of rational numbers, then putting $d_i = \frac{d^i}{i!}$ we have that $D = (d_i)_{i\in\mathbb{N}}$ is a higher derivation of R. Thus the derivation d is algebraic over R if and only if there exists a relation $\sum_{i=0}^{n} r_i d_i(x) = 0$, for all $x \in R$ and some elements $r_0, r_1, \ldots, r_n \neq 0$ in R.

There is another situation in which higher derivations satisfying a linear relation naturally appear. Assume that R is an algebra of finite dimension over a field K. Then every HD $D = (d_i)_{i\in\mathbb{N}}$, where d_i is K-linear for every i, satisfies a K-linear relation and then an R-linear relation, since $\mathrm{End}_K(R)$ is of finite dimension over K.

Consequently, it is natural to study higher derivations which satisfy linear relations on a semiprime ring R. This is the purpose of our paper.

Assume that R is a semiprime ring and $D = (d_i)_{i\in\mathbb{N}}$ is a higher derivation of R. Then there is a unique extension $D^\star = (d_i^\star)_{i\in\mathbb{N}}$ of D to a higher derivation of Q. One of the main results of this paper states that the following conditions are equivalent: there exists an R-linear (Q-linear) relation for D on R; there exists an R-linear (Q-linear) relation for $D^\star$ on Q. This is an extension of the result mentioned above. The corresponding relation with C-algebraicity is no more true in this case.

Also, if R is a prime ring and $D = (d_i)_{i\in\mathbb{N}}$ is a higher derivation of R which satisfies a minimal linear relation of the type $\sum_{i=0}^{n} r_i d_i(x) = 0$, for all $x \in R$, where $r_i \in R$ and $r_n \neq 0$, we give a description of $d_1, d_2, \ldots, d_n$ in terms of X-inner derivations of R. This result is an extension of the result which states that an algebraic derivation is X-inner, provided that $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) = p$ is greater than the degree of algebraicity of d.

In the first section of the paper we prove the main results mentioned above. Some additional results and examples are given in Sec. 2.

Throughout this paper T denotes an arbitrary ring and by R we always denote a semiprime ring, not necessarily with an identity element. Also, Q denotes the left Martindale ring quotients of R and C the center of Q, i.e., the extended centroid of R. Thus when R is a prime ring, C is a field. Finally, $\mathbb{N}$ is the set of natural numbers including 0.

1. MAIN RESULTS

Recall that a family of additive mappings $D = (d_i)_{i\in\mathbb{N}}$ of a ring T is said to be a higher derivation (HD, for short) if $d_0 = id_T$ and for every $n \in \mathbb{N}$ we have $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$, for all $a, b \in R$ ([5], Exercise 4, p. 532).

Assume that $D = (d_i)_{i\in\mathbb{N}}$ is a HD of a semiprime ring R. Then D can be extended in a unique way to a HD $D^\star = (d_i^\star)_{i\in\mathbb{N}}$ of Q. This result is an easy extension of a well-known result for a derivation d, but we could not find it in the literature. So we include a sketch of the proof for the sake of completeness.

Proposition 1.1. *Let R be a semiprime ring and $D = (d_i)_{i\in\mathbb{N}}$ a HD of R. Then there exists a unique HD $D^\star = (d_i^\star)_{i\in\mathbb{N}}$ of Q such that $d_n^\star|_R = d_n$, for every $n \in \mathbb{N}$.*

Proof. Note that if I is and ideal of R, then $d_{n-i}(I^{n+1}) \subseteq I^{i+1}$, for any $i \leq n$ in $\mathbb{N}$.

We define $d_n^\star(q)$ by induction. Put $d_0^\star = id_Q$. Assume that $q \in Q$ and take an essential ideal I of R such that $Iq \subseteq R$. Suppose that $d_i^\star(q)$ has been defined and $I^{i+1}d_i^\star(q) \subseteq R$, for all $i < n$. Then we put

$$ad_n^\star(q) = d_n(aq) - \sum_{i=0}^{n-1} d_{n-i}(a)d_i^\star(q),$$

for every $a \in I^{n+1}$. Now is easy to complete the proof. ∎

Assume that $D = (d_i)_{i\in\mathbb{N}}$ is a HD of a ring T and S is a ring such that either $S \subseteq T$ or $T \subseteq S$.

Definition 1.2. *We say that D satisfies an S-linear relation on T if there exist $a_0, a_1, \ldots, a_n \neq 0$ in S such that $\sum_{i=0}^{n} a_i d_i(x) = 0$, for every $x \in T$.*

In this case we will simply write *D satisfies an S-LR on T*, for short. The above relation will be written as $\sum_{i=0}^{n} a_i d_i = 0$. The integer n is called the length of the relation. If n is minimal among the lengths of all the relations, then the relation is said to be a minimal relation and the integer n is said to be the S-length of D on T, or simply the length of D if there is no possibility of misunderstanding.

If the ring S has an identity element and in the relation above we have $a_n = 1$, then the relation is said to be a monic relation. The length of a monic relation and the monic length of D are defined similarly.

Recall that a derivation $d: R \to R$ is said to be inner if there exists $a \in R$ such that $d(x) = xa - ax$, for every $x \in R$. This derivation is usually called the inner derivation adjoint to $-a \in R$ and will be denoted here by δ_a. Also, if $d : R \to R$ is a derivation of R and there exists $q \in Q$ such that $d(x) = \delta_q(x)$, for every $x \in R$, then d is said to be X-inner [6]. In this case we will denote d again by δ_q.

If I is an ideal of the semiprime ring R and $A(I)$ is the annihilator of I in R, then the ideal $I \oplus A(I)$ is an essential ideal of R. Thus the mapping f : $I \oplus A(I) \to R$ defined by $f(a+b) = a$, for every $a \in I$ and $b \in A(I)$, defines an idempotent element of the extended centroid of R. In the following we denote this idempotent by e_I. For $a \in I$ and $b \in A(I)$ we have $e_I(a+b) = (a+b)e_I = a$.

Theorem 1.3. *Assume that R is a semiprime ring and $D = (d_i)_{i\in\mathbb{N}}$ is a HD of R which satisfies an R-linear relation of minimal length n on R. Then there exist $I \lhd R$ and $q_0 = e_I, q_1, \ldots, q_{n-1} \in e_I Q$ such that $\sum_{i=1}^{n} q_{n-i} d_i(x) = 0$, for all $x \in R$. Moreover, the relation $\sum_{i=1}^{n} q_{n-i} d_i = 0$ on R is minimal, $e_I d_1 = \delta_{q_1}$ and*

$$e_I d_m(x) = \delta_{q_m}(x) - \sum_{i=1}^{m-1} q_i d_{m-i}(x),$$

for every $x \in R$ and $2 \leq m < n$.

Proof. Denote by I the set of all the elements $a \in R$ such that there exist $a_0, a_1 \ldots, a_{n-1} \in R$ with $a d_n(x) + \sum_{i=0}^{n-1} a_i d_i(x) = 0$, for all $x \in R$. Then I is a non-zero left ideal of R. Also, if $a \in I$ and $b \in R$, from $\sum_{i=0}^{n} a_i d_i(bx) = 0$, where $a_n = a$, it follows that $\sum_{j=0}^{n} (\sum_{k=0}^{n-j} a_{j+k} d_k(b)) d_j(x) = 0$ and hence $ab \in I$. Consequently, I is an ideal of R.

By the minimality of n we have that for $a \in I$ the elements $a_0, a_1, \ldots, a_{n-1}$ of R such that $\sum_{i=0}^{n} a_i d_i(x) = 0$, for every $x \in R$, are uniquely determined, where $a_n = a$. Thus the mappings $\varphi_i : I \oplus A(I) \to R$ given by $\varphi_i(a+b) = a_i$, for $a \in I$ and $b \in A(I)$, are well-defined left R-homomorphisms. Therefore there exist $q_0 = e_I, q_1, \ldots, q_n \in e_I Q$ such that $a_i = \varphi_i(a+b) = (a+b) q_{n-i}$, for every $a \in I$, $b \in A(I)$ and $i = 0, 1, \ldots, n$. Also we have

$$(a+b)(e_I d_n(x) + \sum_{j=0}^{n-1} q_{n-j} d_j(x)) = a d_n(x) + \sum_{j=0}^{n-1} a_j d_j(x) = 0.$$

Consequently $e_I d_n(x) + \sum_{j=0}^{n-1} q_{n-j} d_j(x) = 0$, for every $x \in R$, since R is a semiprime ring and $I \oplus A(I)$ is an essential ideal of R.

Note that the above relation is a Q-linear relation of minimal length for D. In fact, assume that $\sum_{i=0}^{m} p_i d_i(x) = 0$, for all $x \in R$, where $p_i \in Q$, $p_m \neq 0$ and $m < n$. Take an essential ideal H of R such that $H p_i \subseteq R$, for all i, and choose an element $b \in H$ with $bp_m \neq 0$. Then $\sum_{i=0}^{m} bp_i d_i(x) = 0$, for every $x \in R$, which contradicts the minimality of n.

For $x, y \in R$ we have $0 = \sum_{j=0}^{n} q_{n-j} d_j(yx) = \sum_{i=0}^{n}(\sum_{k=0}^{n-i} q_{n-i-k} d_k(y))$ $d_i(x)$. Also $\sum_{i=0}^{n} yq_{n-i} d_i(x) = 0$. Substracting both relations and using the minimality of n we obtain $\sum_{k=0}^{n-i} q_{n-i-k} d_k(y) = yq_{n-i}$, for all $y \in R$ and $i = 0, 1, \ldots, n-1$.

In particular, for $i = n-1$ we get $q_1 y + q_0 d_1(y) = yq_1$ and thus $e_I d_1 = \delta_{q_1}$. Also, for any $2 \leq s \leq n$ we have $e_I d_s(y) = \delta_{q_s}(y) - \sum_{i=1}^{s-1} q_i d_{s-i}(y)$, for every $y \in R$. Finally, this implies that $yq_n = e_I d_n(y) + \sum_{k=0}^{n-1} q_{n-k} d_k(y) = 0$ and consequently $q_n = 0$. The proof is complete. ■

As a particular case of Theorem 1.3 we have the following corollary which gives the expression of $(d_i)_{1 \leq i \leq n}$ in terms of inner derivations of Q when R is prime. This result can be considered as an extension of the Kharchenko's result on algebraic derivations of a prime ring already mentioned in the introduction [6].

Corollary 1.4. *Assume that R is a prime ring and $D = (d_i)_{i \in \mathbb{N}}$ is a HD of R which satisfies an R-linear relation of minimal length n on R. Then there exist $q_0 = 1, q_1, \ldots, q_{n-1} \in Q$ such that $\sum_{i=1}^{n} q_{n-i} d_i = 0$. In particular, D satisfies a monic Q-linear relation of length n on R. Moreover, $d_1 = \delta_{q_1}$ and $d_m(x) = \delta_{q_m}(x) - \sum_{i=1}^{m-1} q_i d_{m-i}(x)$, for every $x \in R$ and $2 \leq m \leq n$.*

Remark 1.5. The relations in Corollary 1.4 can be written using matrices. In fact, for every $m \leq n$ we have

$$\begin{bmatrix} \delta_{q_1} \\ \delta_{q_2} \\ \vdots \\ \delta_{q_m} \end{bmatrix} = A(m,q) \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix},$$

where $q = (q_1, ..., q_{n-1})$ and the matrix $A(m,q)$ is the $m \times m$ matrix over Q defined by

$$A(m,q) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ q_1 & 1 & 0 & \ldots & 0 & 0 \\ q_2 & q_1 & 1 & \ldots & 0 & 0 \\ q_3 & q_2 & q_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ q_{m-1} & q_{m-2} & q_{m-3} & \ldots & q_1 & 1 \end{bmatrix}.$$

It is clear that $A(m,q)$ is invertible for any $m \leq n$ and so

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} = A(m,q)^{-1} \begin{bmatrix} \delta_{q_1} \\ \delta_{q_2} \\ \vdots \\ \delta_{q_m} \end{bmatrix},$$

where $A(m,q)^{-1} = (b_{ij})_{m\times m}$ is given by

$$\begin{cases} b_{ii} = 1, \\ b_{ij} = 0, \quad \text{if } i < j, \\ b_{l+i,i} = b_{l+1,1} = -\sum_{j=1}^{l} b_{l+1-j,1} q_j, \quad \text{for } 1 \le l \le m-1. \end{cases}$$

■

Now we prove the following lemma which is related to Corollary 1.4.

Lemma 1.6. *Assume that T is a ring, $a_1, a_2, \ldots \in T$ and $D = (d_i)_{i\in\mathbb{N}}$ is defined by induction as $d_0 = id_T$, $d_1 = \delta_{a_1}$, and $d_m(x) = \delta_{a_m}(x) - \sum_{i=1}^{m-1} a_i d_{m-i}(x)$, for every $x \in T$ and $m \ge 2$. Then D is a HD of T. Furthermore, if a_n is a linear combination of $\{1, a_1, \ldots, a_{n-1}\}$ with coefficients in $Z(T)$, the center of T, then D satisfies a monic T-linear relation on T of length n.*

Proof. It is clear that d_i is an additive mapping, for all i, and d_1 is a derivation of T. By induction we assume that $d_s(xy) = \sum_{i=0}^{s} d_i(x)d_{s-i}(y)$, for all $x, y \in R$ and $s < m$. Then we have

$$\begin{aligned} d_m(xy) &= \delta_{a_m}(xy) - \sum_{i=1}^{m-1} a_i d_{m-i}(xy) \\ &= \delta_{a_m}(x)y + x\delta_{a_m}(y) - \sum_{i=1}^{m-1} a_i \sum_{j=0}^{m-i} d_j(x) d_{m-i-j}(y) \\ &\quad + \sum_{l=1}^{m-1} d_l(x)d_{m-l}(y) - d_1(x)d_{m-1}(y) \\ &\quad - \sum_{l=2}^{m-1}\left(xa_l - a_l x - \sum_{s=1}^{l-1} a_s d_{l-s}(x)\right) d_{m-l}(y) \\ &= x(\delta_{a_m}(y) - a_1 d_{m-1}(y) - \sum_{l=2}^{m-1} a_l d_{m-l}(y)) \\ &\quad + \sum_{l=1}^{m-1} d_l(x)d_{m-l}(y) + \left(\delta_{a_m}(x) - \sum_{i=1}^{m-1} a_i d_{m-i}(x)\right) y \\ &\quad - \sum_{i=1}^{m-1} a_i \sum_{j=0}^{m-i-1} d_j(x) d_{m-i-j}(y) \\ &\quad + \sum_{l=2}^{m-1}\left(a_l x + \sum_{s=1}^{l-1} a_s d_{l-s}(x)\right) d_{m-l}(y) + a_1 x d_{m-1}(y) \end{aligned}$$

$$
\begin{aligned}
&= xd_m(y) + \sum_{l=1}^{m-1} d_l(x)d_{m-l}(y) + d_m(x)y - a_{m-1}xd_1(y) \\
&\quad + a_1xd_{m-1}(y) - \sum_{i=1}^{m-2} a_i \sum_{j=0}^{m-i-1} d_j(x)d_{m-i-j}(y) \\
&\quad + \sum_{l=2}^{m-1}\left(a_lx + \sum_{s=1}^{l-1} a_sd_{l-s}(x)\right)d_{m-l}(y) \\
&= \sum_{l=0}^{m} d_l(x)d_{m-l}(y) - \sum_{l=2}^{m-1}\sum_{i=1}^{l-1} a_id_{l-i}(x)d_{m-l}(y) \\
&\quad + \sum_{l=2}^{m-1}\sum_{s=1}^{l-1} a_sd_{l-s}(x)d_{m-l}(y) = \sum_{l=0}^{m} d_l(x)d_{m-l}(y).
\end{aligned}
$$

Therefore, $d_n(xy) = \sum_{i=0}^{n} d_i(x)d_{n-i}(y)$ for every $x, y \in T$ and for all $n \geq 1$. The first part follows.

Now assume that $a_n = \sum_{j=1}^{n-1} a_j\lambda_j + \lambda$, where $\lambda, \lambda_j \in Z(T)$. Then $\delta_{a_n} = \sum_{j=1}^{n-1} \delta_{a_j}\lambda_j$, and so $d_n(x) = -\sum_{i=1}^{n-1} a_id_{n-i}(x) + \sum_{j=1}^{n-1} \delta_{a_j}\lambda_j$. Note that by the definition of D every δ_{a_j} is a linear combination of d_i with coefficients in T for $i \leq j$. The result easily follows. ■

Remark 1.7. By Lemma 1.6, for the elements $q_1, \ldots, q_{n-1}$ of Q given in Corollary 1.4 we can define mappings from Q to Q by $g_1 = \delta_{q_1}$ and $g_m(q) = \delta_{q_m}(q) - \sum_{i=1}^{m-1} q_ig_{m-i}(q)$, for every $q \in Q$ and $2 \leq m \leq n$, where $q_n = 0$. Hence Corollary 1.4 implies that these are the unique extensions $d_i^\star$ of d_i to Q for $i \leq n$. More generally, the expressions obtained in Theorem 1.3 for e_Id_m define in a similar way the unique extensions $d_m^\star$ in e_IQ, $1 \leq m \leq n$.

Note that if $D = (d_i)_{i\in\mathbb{N}}$ is a HD of a ring T and e is a central idempotent of T, then $d_i(e) = 0$ for every $i \geq 1$. In fact, if $d_i(e) = 0$ for $1 \leq i < n$, then $d_n(e) = d_n(e^2) = 2d_n(e)e$. Hence $d_n(e)e = 2d_n(e)e$, so $d_n(e)e = 0$ and it follows that $d_n(e) = 0$. We use this in the next result.

Now we are in a position to prove one of the main results of this paper. If D is a HD of the semiprime ring R, then we denote by $D^\star = (d_i^\star)_{i\in\mathbb{N}}$ the unique extension of D to Q. We have:

Theorem 1.8. *Assume that R is a semiprime ring and D is a HD of R. Then the following conditions are equivalent:*

(i) D satisfies an R-LR on R;
(ii) D satisfies a Q-LR on R;
(iii) $D^\star$ satisfies a Q-LR on Q;
(iv) $D^\star$ satisfies an R-LR on Q.

Furthermore, if one of these equivalent conditions holds, then the minimal lengths of the relations is always the same.

Proof. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii) are tautologies. Also, implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) follow from the fact that every Q-LR on T can be reduced, by multiplying by a suitable element from R to an R-LR on T, for $T = R, Q$, respectively. It remains to prove (i) $\Rightarrow$ (iii).

Assume that $\sum_{i=0}^{n} r_i d_i = 0$, $r_i \in R$, $r_n \neq 0$ is a minimal relation for D on R. So by Theorem 1.3 there exist a non-zero ideal I of R and $q_i \in e_I Q$, $0 \leq i \leq n-1$, such that $\sum_{i=1}^{n} q_{n-i} d_i(x) = 0$, for every $x \in R$, where $q_0 = e_I$. Moreover, $e_I d_1 = \delta_{q_1}$ and $e_I d_m = \delta_{q_m} - \sum_{i=1}^{m-1} q_i d_{m-i}$, for $1 \leq m < n$.

By the above remark we have that $d_i^{\star}(e_I R) \subseteq e_I R$, for every $i \in \mathbb{N}$, where $e_I R$ is a semiprime ring having $e_I Q$ as left Martindale ring of quotients.

On the other hand, choose any elements $q_{n+1}, q_{n+2}, \ldots \in e_I Q$ and put $q_n = 0$. By Lemma 1.6 the family of mappings $G = (g_i)_{i \in \mathbb{N}}$ of $e_I Q$ defined by $g_0 = id_{e_I Q}$, $g_1 = \delta_{q_1}$ and $g_m = \delta_{q_m} - \sum_{i=1}^{m-1} q_i g_{m-i}$, for $m \geq 1$, defines a HD of $e_I Q$ such that

$$e_I g_n = \delta_{q_n} - \sum_{i=1}^{n-1} q_i g_{n-i} = -\sum_{i=1}^{n-1} q_i g_{n-i}.$$

Therefore G satisfies a Q-linear relation on $e_I Q$.

Now extend G to a HD of Q by defining $G\,|_{(1-e_I)Q} = D^{\star}\,|_{(1-e_I)Q}$. Since $g_i\,|_R = d_i$, for $0 \leq i \leq n$, it follows that $g_i = d_i^{\star}$ and so $D^{\star}$ satisfies a Q-linear relation of length n on $e_I Q$. However, note that the above relation is obviously satisfied on $(1-e_I)Q$ and thus on Q. The proof is complete since the rest is clear. ■

Remark 1.9. If R is a prime ring and D is a HD of R which satisfies an R-linear relation of minimal length n on R, then by Corollary 1.4 D satisfies a monic Q-linear relation on R of the same length n. The converse is clearly true. Thus in this case we can add another equivalent condition in the statement of Theorem 1.8: *D satisfies a monic Q-LR on R*. Furthermore, the length of $D^{\star}$ on Q concides with the monic length of $D^{\star}$.

2. ADDITIONAL REMARKS

It is clear that if $D = (d_i)_{i \in \mathbb{N}}$ satisfies a C-linear relation, then D satisfies a Q-linear relation. For a derivation d of a semiprime ring R it is proved that d is algebraic over R if and only if it is algebraic over C ([8],

[13]). It is a natural question to ask whether a similar result holds for a HD. The following example shows that this is not the case.

First note that for any ring T and $a \in T$, the sequence $D = (d_i)_{i \in \mathbb{N}}$ defined by $d_0(t) = t$, $d_i(t) = (-1)^i(a^i t - a^{i-1}ta)$, for every $t \in T$, $i \geq 1$, is a HD of T (cf. [12]).

Example 2.1. Let $R = K\langle x, y\rangle$ denote the free ring in the indeterminates x and y over a field K. Define as above $D = (d_i)_{i \in \mathbb{N}}$ by $d_0(r) = r$, $d_i(r) = (-1)^i(y^i r - y^{i-1}ry)$, for every $r \in R$, $i \geq 1$. Then $d_2(r) + yd_1(r) = 0$, for every $r \in R$ and so D satisfies an R-LR of length 2 on R. Nevertheless D does not satisfy a C-LR on R, where C is the extended centroid of R.

In fact, recall that $C = K$ ([14], Theorem 2.5). If for $c_i \in K$, $0 \leq i \leq n-1$, we have $d_n + \sum_{i=0}^{n-1} c_i d_i = 0$, then

$$(-1)^n(y^n r - y^{n-1}ry) + \sum_{i=1}^{n-1}(-1)^i c_i(y^i r - y^{i-1}ry) + c_0 r = 0,$$

for every $r \in R$. In particular, for $r = x$ we obtain

$$(-1)^n y^n x = (-1)^{n+1} y^{n-1}xy - \sum_{i=1}^{n-1}(-1)^i c_i(y^i x - y^{i-1}xy) - c_0 x,$$

which is clearly a contradiction. ∎

For a derivation d of a prime ring R it is known that if d is algebraic over an ideal I of R, then d is algebraic over R ([8], Théorème 1.9). The same result holds for higher derivations. More generally we have:

Theorem 2.2. *Let $D = (d_i)_{i \in \mathbb{N}}$ be a HD of a semiprime ring R and I denote an essential ideal of R. If $D|_I = (d_i|_I)_{i \in \mathbb{N}}$ satisfies an R-linear relation on I, then D satisfies an R-linear relation on R.*

Proof. It is known that I is a semiprime ring and the left Martindale ring of quotients of I is equal to Q. Assume that $\sum_{i=0}^{n} r_i d_i(a) = 0$ for every $a \in I$, where $r_i \in R$ and $r_n \neq 0$. Thus $I(\sum_{i=0}^{n} r_i d_i(a)) = 0$, for every $a \in I$, so we may assume that the relation has coefficients in I. Take a relation of this type of minimal length and argue as in Theorem 1.3. So we can find an idempotent e of C and elements $q_0 = e, q_1, q_2, \ldots, q_{n-1} \in eQ$ such that $\sum_{i=1}^{n} q_{n-i}d_i = 0$, $ed_1 = \delta_{q_1}$ and $ed_m(x) = \delta_{q_m}(x) - \sum_{i=1}^{m-1} q_{m-i}d_i(x)$, for every $x \in I$ and $m < n$.

Now define $g_m : Q \to Q$, for $0 \leq m \leq n$, by $g_0 = id_Q$, $g_1 = \delta_{q_1}$ and $g_m = \delta_{q_m} - \sum_{i=1}^{m-1} q_{m-i}d_i$, where $q_n = 0$. By Lemma 1.6, we can complete $G = (g_i)_{0 \leq i \leq n}$ to a HD of Q by choosing arbitrary elements $q_{n+1}, \ldots$. In this way we have a HD $G = (g_i)_{i \in \mathbb{N}}$ of Q such that $eg_n + \sum_{j=1}^{n-1} q_{n-j}g_j = 0$ in Q.

Note that $g_i \mid_{eI} = d_i \mid_{eI}$, for $0 \leq i \leq n$. Since the extension of a HD of eI to the quotient ring eQ is unique, we obtain $g_i \mid_{eQ} = d_i^\star \mid_{eQ}$, for $0 \leq i \leq n$. Hence $ed_n^\star(x) + \sum_{j=1}^{n-1} q_{n-j} d_j^\star(x) = 0$, for every $x \in eR$. Since $d_j^\star((1-e)R) \subseteq (1-e)Q$ for all j, the last relation holds also for $x \in (1-e)R$. Consequently $ed_n(x) + \sum_{j=1}^{n-1} q_{n-j} d_j(x) = 0$, for every $x \in R$. This completes the proof. ■

Example 2.3. Of course we can have monic relations of various lengths. For example, if R is a simple ring with an identity, $Z(R)$ is the center of R, $a \in R \setminus Z(R)$ and $D = (d_i)_{i \in \mathbb{N}}$ is the HD of R defined by $d_0(x) = x$, $d_i(x) = (-1)^i (a^i x - a^{i-1} x a)$, for all $x \in R$, $i \geq 1$, we have the relations $d_2 + ad_1 = 0$, $d_3 + ad_2 = 0$ and $d_3 + (a+c)d_2 + acd_1 = 0$, for any $c \in Z(R)$.

Note that $\{1, a\}$ is linearly independent over C meanwhile the sets $\{1, a, 0\}$ and $\{1, a+c, ac\}$ are linearly dependent. ■

Now we study a criterious for recognizing when a Q-monic relation is minimal, where Q is the left Martindale ring of quotients of a prime ring R.

Theorem 2.4. *Assume that R is prime, $q_1, q_2, \ldots \in Q$ and $D = (d_i)_{i \in \mathbb{N}}$ is the HD of Q defined by $d_0 = id_Q$, $d_1 = \delta_{q_1}$, $d_m = \delta_{q_m} - \sum_{i=1}^{m-1} q_i d_{m-i}$, for all $m \geq 0$. If $d_i(R) \subseteq R$ for every $i \geq 1$, then $D \mid_R = (d_i \mid_R)_{i \in \mathbb{N}}$ is a HD of R. Moreover, in this case $D \mid_R$ satisfies an R-linear relation on R if and only if there exists n such that q_n is a linear combination of $1, q_1, \ldots, q_{n-1}$ with coefficients in C. Finally, the length of $D \mid_R$ is equal to the smallest integer n which satisfies this condition.*

Proof. By Lemma 1.6 and the assumption $D \mid_R$ is a HD of R. Also, the same lemma shows that if there exists an integer n such that q_n is a linear combination of $1, q_1, \ldots, q_{n-1}$ with coefficients in C, then D satisfies a monic Q-linear relation of length n on Q. Thus Theorem 1.8 implies that $D \mid_R$ satisfies an R-linear relation of length $m \leq n$ on R.

Now, assume that $D \mid_R$ satisfies an R-linear relation on R and let s be the length of D. By Corollary 1.4 there exist $p_1, p_2, \ldots, p_{s-1} \in Q$ such that $d_1 = \delta_{p_1}$, $d_m = \delta_{p_m} - \sum_{i=1}^{m-1} p_i d_{m-i}$ for $1 < m \leq s$, where $p_s = 0$.

Note that $\delta_{q_1} = \delta_{p_1}$ and so there exists $c_1 \in C$ such that $q_1 - p_1 = c_1$. We claim that for every $t \leq s$ we have $q_t - p_t - \sum_{j=1}^{t-1} c_j q_{t-j} = c_t$, where $c_t, c_j \in C$. Since $p_s = 0$ this implies that $q_s = \sum_{j=1}^{s-1} c_j q_{s-j} + c_s$ and so the claim completes the proof.

Assume, by induction, that $q_l - p_l - \sum_{j=1}^{l-1} c_j q_{l-j} = c_l$, where $c_l, c_j \in C$, for $l < m \leq s$. We have

$$\begin{aligned} 0 &= \delta_{q_m} - \sum_{i=1}^{m-1} q_i d_{m-i} - \delta_{p_m} + \sum_{i=1}^{m-1} p_i d_{m-i} \\ &= \delta_{q_m} - \delta_{p_m} - \sum_{i=1}^{m-1} d_i (q_{m-i} - p_{m-i}). \end{aligned}$$

Note that by Remark 1.5

$$d_i = \sum_{j=1}^{i} b_{ij}\delta_{q_j} = \sum_{j=1}^{i} b_{i-j+1,1}\delta_{q_j},$$

where $(b_{ij})_{m\times m}$ is the matrix $A(m,q)^{-1}$ defined in the remark. Thus, putting $q_0 = 1$, by the inductive assumption we obtain

$$\begin{aligned}
\delta_{q_m} - \delta_{p_m} &= \sum_{i=1}^{m-1} d_i(q_{m-i} - p_{m-i}) \\
&= \sum_{i=1}^{m-1}\sum_{l=1}^{i} b_{i-l+1,1}\delta_{q_l}\sum_{j=1}^{m-i} c_j q_{m-i-j} \\
&= \sum_{l=1}^{m-1}\sum_{i=l}^{m-1} b_{i-l+1,1}\delta_{q_l}\sum_{j=1}^{m-i} c_j q_{m-i-j} \\
&= \sum_{l=1}^{m-1}\sum_{h=1}^{m-l} b_{h1}\delta_{q_l}\sum_{j=1}^{m-(h+l-1)} c_j q_{m-(h+l-1)-j} \\
&= \sum_{l=1}^{m-1}\sum_{j=1}^{m-l} c_j\delta_{q_l}\sum_{h=1}^{m-(l+j-l)} b_{h1} q_{m-(h+l+j-1)} \\
&= \sum_{l=1}^{m-1} c_{m-l}\delta_{q_l}.
\end{aligned}$$

(The last equality holds because $(b_{ij})A(m,q)$ is the identity matrix).

Hence

$$\delta_{q_m - p_m - \sum_{l=1}^{m-1} c_{m-l}q_l} = 0$$

and consequently there exists $c_m \in C$ such that $q_m - p_m - \sum_{l=1}^{m-1} c_{m-l}q_l = c_m$. The proof is complete. ■

As an immediate consequence of Corollary 1.4, Lemma 1.6 and Theorem 2.4 we have the following:

Corollary 2.5. *Assume that R prime and $D = (d_i)_{i\in\mathbb{N}}$ is a HD of R which satisfies a monic relation $\sum_{i=1}^{n} q_{n-i}d_i = 0$, where $q_i \in Q$ for $1 \le i \le n-1$ and $q_0 = 1$. Then the given relation is minimal if and only if the set $\{1, q_1, q_2, \ldots, q_{n-1}\}$ is linearly independent over C.*

Suppose that R is a prime algebra over the field of rational numbers and d is an algebraic derivation of R. Then d is X-inner and $D = (d_i)_{i\in\mathbb{N}}$

defined by $d_i = \frac{d^i}{i!}$, for $i \in \mathbb{N}$, is a HD of R. Since d is algebraic over R, D satisfies an R-linear relation. So by Corollary 1.4 the first terms of D are determined by inner derivations of Q. The following theorem gives the precise expression of d_m, for all m. The proof of this result is done by induction and requires tedious computations. So we will omit it here.

Theorem 2.6. *Assume that R is an algebra over the field of rational numbers and d is the inner derivation adjoint to $-a \in R$. Then for every $m \geq 2$ we have*

$$d_m(x) = \frac{1}{\left(\prod_{l=1}^{m-1}(m-l)!\right)^2} \sum_{i=0}^{m-1} (-1)^i \frac{a^i}{(m-i)!i!} \delta_{a^{m-i}}(x),$$

for every $x \in R$, where $d_m = \frac{d^m}{m!}$.

Remark 2.7. When R is a prime algebra of characteristic p, then the formula from the above theorem holds for any $2 \leq m < p$. Using Theorem 2.6, Theorem 2.4 and Corollary 2.5 one can easily get another proof of a well-known result which states that if R is a prime ring and d is the X-inner derivation of R adjoint to $q \in Q$, then d is algebraic over R of degree $n \leq p$ if and only if q is algebraic over C of degree n ([8], Lemma 1.4). Thus Theorem 2.4 can be considered as an extension of this last result.

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