II Encuentro Nacional de Algebra

The History of the Higher Order Derivations in Rings and Some Related Problems

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Introduction

We consider here not necessarily commutative rings R and they not necessarily have identity elements.

Definition 1 *A map* $d\colon R \to R$ *is a derivation if* d *is additive and* $d(ab) = d(a)b + ad(b)$*, for all* $a, b \in R$*.*

Rings with derivations have been studied in many papers in the last 50 years. Many questions have been considered.

Relationships between derivations and the structure of rings.

Powers (or products) of derivations and commutativity of rings (Posner's theorems).

Other results: Jensen, Creedon

They were already generalized in several directions.

Posner's Theorems (1957):

1^{st}: d_1, d_2 derivations, R prime, $char(R) \neq 2$, $d_1 \cdot d_2$ derivation $\Rightarrow d_1 = 0$ or $d_2 = 0$.

2^{nd}: d derivation, R prime, $[x, d(x)] \in Z(R)$, $\forall x \in R$ $\Rightarrow d = 0$ or R is commutative.

Latelly, in 1998, T. Creedon extended the Posner's First Theorem to semiprime algebras, proving that if the product of two derivations in an algebra A is a derivation, then the product maps the algebra into the nil radical $nil(A)$ (the intersection of all prime ideals of A). Thus, if the product of two derivations in a semiprime algebra is a derivation, then the product is zero.

Moreover, Creedon obtained conditions proving that the product of two derivations maps the algebra into the Jacobson radical.

These problemas didn't have been studied in the HD context.

- **I.N. Herstein**, *“Lie and Jordan structures in simple, associative rings”*, Bull. Amer. Math. Soc. 67 (1961), 517-531
- **I.N. Herstein**, *“Topics in ring theory”*, Univ. of Chicago Press, Chicago, 1969

Herstein's Theorem for Derivations

Definition 2 *An additive mapping* $d : R \to R$ *is said to be a Jordan derivation if* $d(a^2) = d(a)a + ad(a)$ *is fulfilled for all* $a \in R$.

Theorem 3 *(Herstein) Assume that* R *is a prime ring of* $char(R) \neq 2$. *Then any Jordan derivation of* R *is a derivation.*

(The result is no loger true if $char(R) = 2$, the converse is obviously true.)

M. Brešar[a] extended this result to semiprime rings.

Later on Brešar[b] studied triple derivations.

Recall that $d : R \to R$ is said to be a Jordan triple derivation (JTD, for short) of R if

$$d(aba) = d(a)ba + ad(b)a + abd(a),\ \forall a, b \in R.$$

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[a] *"Jordan derivations on semiprime rings"*, Proc. AMS 104 (4) (1988), 1003-1006

[b] *"Jordan mappings of semiprime rings"*, J. Algebra 127 (1989), 218-228

He gave another proof of Herstein's Theorem for semiprime rings showing that

Theorem 4 *Let R be a 2-torsion-free semiprime ring and d a Jordan triple derivation of R. Then d is a derivation of R.*

It turns out that every Jordan derivation of a 2-torsion-free ring is a Jordan triple derivation.

So, Brešar's result proves that any Jordan derivation is a derivation (for semiprime rings with no 2-torsion-free elements).

If R is a ring, R has a Lie structure by the bracket product $[x, y] = xy - yx$, for $x, y \in R$.

Following C. Lanski and S. Montgomery[a], a *Lie ideal* of R is any additive subgroup U of R with $[u, r] \in U$ for all $u \in U$ and $r \in R$.

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[a] *"Lie structure of prime rings of characteristic* 2*"*, Pacific J. of Math. 42 (1) (1972), 117-136

Also, R. Awtar extended the Herstein's Theorem to Lie ideals[a] by proving that if U is a Lie ideal of a prime ring R of $char(R) \neq 2$ such that $u^2 \in U$, for every $u \in U$, and d: $R \to R$ is an additive mapping such that $d\mid_U$ is a Jordan derivation of U into R, then $d\mid_U$ is a derivation of U into R, as we will clear latelly.

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[a] *"Lie ideals and Jordan derivations of prime rings"*, Proc. AMS 90 (1) (1984), 9-14

Following B. Hvala[a], an additive mapping $F\colon R \to R$ is called a *generalized derivation* if there exists a derivation $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

We call an additive mapping $F: R \to R$ a *Jordan generalized derivation* if there exists a derivation $d\colon R \to R$ such that $F(x^2) = F(x)x + xd(x)$ holds for all $x \in R$.

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[a] *"Generalized derivations in rings"*, Comm. Algebra 26 (4) (1988), 1147-1166

M. Ashraf and N. Rehman[a] showed that in a 2-torsion-free ring R which has a commutator nonzero divisor, every Jordan generalized derivation on R is a generalized derivation.

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[a] *"On Jordan generalized derivations in rings"*, Math. J. Okayama Univ. 42 (2000), 7-9

Generalized derivations have been studied by analysts in the context of algebras on certain normed spaces.

By a generalized derivation of an algebra A one usually means a map of the form $x \rightsquigarrow ax + xb$, where a and b are fixed elements in A.

Hvala called such maps *generalized inner* (or *inner generalized*) derivations for they present a generalization of the concept of inner derivations:

$$f(x) = ax + xb \Rightarrow f(xy) = f(x)y + xd_b(y),$$

where $d_b(y) = yb - by$ is an inner derivation.

In the theory of operator algebras, they are considered as an important class of the so-called elementary operators, that is, operators where

$$x \rightsquigarrow \sum_{i=1}^{n} a_i x b_i.$$

Hvala's concept of generalized derivation covers both concepts of derivations and of generalized inner derivations.

Moreover, generalized derivations with $d = 0$ cover the concept of left multiplication:

$$f(xy) = f(x)y, \quad \forall x, y \in R.$$

Higher Derivations

Assume that R is an algebra over the rational field $\mathbb{Q}$ and d: $R \to R$ is a derivation. Then, if we put $d_n(x) = \frac{d^n(x)}{n!}$ we have that relation:

$$\forall n \geq 1,\ a, b \in R,\ d_n(ab) = \underset{i+j=n}{} d_i(a)d_j(b). \tag{1}$$

So d defines a sequence $d_0, d_1, \ldots, d_n, \ldots$ such that $d_0 = id_R$, d_1 is a derivation and (1) holds.

Definition 5 *A sequence of additive mappings* $D = \{d_0, d_1, \ldots, d_n, \ldots\}$ *is said to be a higher derivation (HD, for short) of R if the above relation (1) holds.*

Note 6 *HD have been considered for the first time in 1936 by H. Hasse and F.K. Schmidt*[a] *and independently by O. Teichmüller*[b]*.*

Note 7 *HD have been considered and applied mainly in commutative rings in many papers, but no so much in noncommutative rings.*

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[a] *"Theorie der höheren Differentiale in einem algebraischen Funktionenkörper mit vollkommenem Konstantenkörper bei beliebiger Charakteristik"*, J. Reine Angew. Math. (Journal für Mathematik) 175 (1936), p. 50-54

[b] *"Differentialrechnung bei Charakteristik p"*, J. Reine Angew. Math. (Journal für Mathematik) 175 (1936), p. 89-99

Examples of Higher Derivations

Example 8 *Let* $\mathbb{K}$ *be any field and define in* $\mathbb{K}[x]$,

$$d_i(x^j) = \binom{j}{i} x^{j-i}, \text{ where } \binom{j}{i} = 0 \text{ if } i > j.$$

Then $D = (d_i)_{i \geq 0}$ *is a HD of* $\mathbb{K}[x]$.

Example 9 *Let* $R = \mathbb{K}[x]\langle y, z\rangle$ *(*$\mathbb{K}$ *field,* $char(\mathbb{K}) \neq 2$*), a 2-torsion-free prime ring. Thus* $Z(R) = \mathbb{K}[x]$*. Let* $D = (d_n)_{n\in\mathbb{N}}$ *be a sequence of additive mappings of* R *such that* $d_0 = id_R$ *and, for every* $n \geq 1$*,*

$$\begin{array}{rcl} d_n(x) & = & p_n(x), \text{ where } p_n(x) \in \mathbb{K}[x]; \\ d_n(y) & = & q_n(x,y,z), \text{ where } q_n(x,y,z) \in R; \\ d_n(z) & = & r_n(x,y,z), \text{ where } r_n(x,y,z) \in R; \\ d_n(a^j) & = & \sum_{i=0}^{n} d_i(a^{j-1})d_{n-i}(a), \ \forall j \geq 2, \end{array}$$

where $a = x$*,* y*, or* z*. Then* D *is a HD of* R*.*

Reciprocally, every HD of R *is of this type for some* $p_i(x)$*,* $q_i(x,y,z)$*,* $r_i(x,y,z)$*.*

There is an interesting relation between **commuting** finite HD and commuting derivations due to D.R. Malm[a].

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[a] *“Simplicity of Partial and Schmidt Differential Operator Rings”*, Pacific J. of Math. 132 (1) (1988), p. 85-112

Example 10 *Let $k \in \mathbb{N}$. Given derivations $d_1, \ldots, d_k$, or a finite HD $(d_0, d_1, \ldots, d_k)$, and $i = 1, \ldots, k$, we put* $d_{(i,k)} = \underset{j(1)+\ldots+j(i)=k}{} d_{j(1)} \ldots d_{j(i)}$.

Assume that $k!$ is invertible in R.

(a). If $\delta_1, \ldots, \delta_k$ are (commuting) derivations, then the rules $d_0 = id_R$, $d_r = \sum_{i=1}^{r} \frac{\delta_{(i,r)}}{i!}$ define a (commuting) HD $(d_0, d_1, \ldots, d_k)$ on R;

(b). If $(d_0, d_1, \ldots, d_k)$ is a (commuting) HD on R, then the rules $\delta_r = \displaystyle\sum_{j=1}^{r} \frac{(-1)^{j+1} d_{(j,r)}}{j}$ define (commuting) derivations $\delta_1, \ldots, \delta_k$ on R.

These processes are inverse each to the other.

Herstein's Theorem for Higher Derivations

Based on Brešar and on Awtar we extended Herstein's theorem for HD.

- **C. Haetinger**, *"Derivações de ordem superior em anéis primos e semiprimos"*, Ph.D. thesis, Universidade Federal do Rio Grande do Sul: Brasil (2000), 1-64.
- **M. Ferrero, C. Haetinger**, *"Higher derivations and a theorem by Herstein"*, Quaest. Math. 25 (2002), 1-9.
- **C. Haetinger**, *"Higher derivations on Lie ideals"*. Seleta do XXIV CNMAC (SBMAC - Brasil). Série Tendências em Matemática Aplicada e Computacional, vol. 3, parte 1, no. (1). 141-145 (2002).

Definition 11 *Let $D = (d_i)_{i \in \mathbb{N}}$ be given. Then D is called:*

1. a Jordan higher derivation (HJD, for short) if for any $n \geq 0$ we have

$$d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a), \ \forall a \in R;$$

2. a Jordan triple higher derivation (JTHD, for short) if for any $n \geq 0$ and $a, b \in R$ we have

$$d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a).$$

Our Results on HD

In the main result we assume that the Lie ideal U is not contained in $Z(R)$ and $u^2 \in U$, $\forall u \in U$.

A Lie ideal of this type will be called an *admissible Lie ideal.*

Similarly as above, if U is a Lie ideal of R, then D is said to be a *HD (JHD, JTHD) of* U *into* R in the case that the above corresponding conditions are satisfied for all $a, b \in U$.

Theorem 12 *Let* R *be a* 2*-torsion-free semiprime ring (resp. prime,* $char(R) \neq 2$ *and* U *an admissible Lie ideal). Then every JTHD of* R *(resp. of* U *into* R*) is a HD of* R *(resp. of* U *into* R*).*

Theorem 13 *Assume that* R *is* 2*-torsion-free and* U *is a Lie ideal of* R *such that* $u^2 \in U, \forall u \in U$*. Then every JHD of* R *(resp. of* U *into* R*) is a JTHD of* R *(resp. of* U *into* R*).*

Corollary 14 *(Herstein's Theorem) Assume that* R *is* 2*-torsion-free and semiprime (resp. prime,* $char(R) \neq 2$ *and* U *an admissible Lie ideal). Then every JHD of* R *(resp. of* U *into* R*) is a HD of* R *(resp. of* U *into* R*).*

One can ask whether the result of Corollary 14 for Lie ideals is also true in the semiprime case. We include in the paper an example by M. Brešar showing that without some additional assumption this is not the case.

The result proved by Awtar also holds trivially for Lie ideals contained in $Z(R)$. We were unable to answer the question on whether the corresponding result is also true for HD.

Generalized Higher Derivations

Definition 15 *Let $F = (f_i)_{i\in\mathbb{N}}$ be a family of additive mappings of R such that $f_0 = id_R$. F is said to be:*

- *a generalized higher derivation*[a] *(GHD, for short) if there exists a HD $D = (d_i)_{i\in\mathbb{N}}$ of R such that for every $n \in \mathbb{N}$ we have*

$$f_n(ab) = \sum_{i+j=n} f_i(a)d_j(b), \ \forall a, b \in R;$$

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[a] **A. Nakajima**, *"On generalized higher derivations"*, Turkish J. of Math. 24 (3) (2000), 295-311

- *a Jordan generalized higher derivation (JGHD, for short) if there exists a HD* $D = (d_i)_{i\in\mathbb{N}}$ *of* R *such that for every* $n \in \mathbb{N}$ *we have*

$$f_n(a^2) = \sum_{i+j=n} f_i(a)d_j(a), \ \forall a \in R;$$

- *a Jordan generalized triple higher derivation (JGTHD, for short) if there exists a HD* $D = (d_i)_{i\in\mathbb{N}}$ *of* R *such that for every* $n \in \mathbb{N}$ *we have*

$$f_n(aba) = \sum_{i+j+k=n} f_i(a)d_j(b)d_k(a), \ \forall a, b \in R.$$

It is clear that in the case of Definition 15, f_1 is a Jordan generalized derivation.

Similarly, if U is a Lie ideal of R, then a family of additive mappings of R, $D = (d_i)_{i\in\mathbb{N}}$ is said to be a *HD (JHD, JTHD) of* U *into* R and a family of additive mappings of R, $F = (f_i)_{i\in\mathbb{N}}$ is said to be a

As we have mentioned above, the main purpose of this lecture[a] is to adress the following result.

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[a] **Cortes, W.; Haetinger, C.**; *"On Jordan generalized higher derivations in rings"*, Turkish Journal of Mathematics 28 (2004), 1-10 (to appear)

Let U be a Lie ideal of a ring R that verifies $u^2 \in U$, for every $u \in U$. A Lie ideal of this type will be called a *square closed Lie ideal.*

Theorem 16 *Let R be a 2-torsion-free ring which has a commutator right nonzero divisor and U a square closed Lie ideal of R. Then every Jordan generalized higher derivation of U into R is a generalized higher derivation of U into R.*

Note 17 *Since $U = R$ is obviously a square closed Lie ideal of R, then Theorem 16 is also true for JGHD of R.*

In particular, if $f_i = d_i$ for every $i \in \mathbb{N}$, we have the following

Corollary 18 *Let R be a 2-torsion-free ring which has a commutator right nonzero divisor and U a square closed Lie ideal of R. Then every JHD of U into R is a HD of U into R.*

Note that Corollary 18 states our result on HD without the semiprimality condition.

We include an example by M. Brešar showing that the semiprimality and the right nonzero divisor commutator assumptions are independent each other.

Furthermore we prove the next theorem, which generalizes our other result on HD.

Theorem 19 *Let* R *be a* 2*-torsion-free ring and* U *a Lie ideal of* R*. Then every Jordan generalized higher derivation of* U *into* R *is a Jordan generalized triple higher derivation of* U *into* R*.*

One can ask whether our result on JTHD is also true for Jordan generalized triple higher derivations.

We were still unable to answer this question.

Note: we could have considered an alternative definition for a JGHD in Definition 15: instead of supposing D as a HD, we could have taking D just as a JHD of R.

In fact, with this assumption, we could prove our main Theorem in the same way as we did.

The following example shows that the assumptions of our Corollary 18 and our Herstein's Theorem on HD (Corollary 14) are independent each other. This example is due to M. Brešar who kindly allowed us to include it here.

Example 20 *A semiprime ring may not contain a commutator nonzero divisor.*

Take commutative semiprime rings, or semiprime rings R containing a nonzero central idempotent element $e \in R$ such that eR is commutative.

Conversely, a ring may contain a commutator nonzero divisor, but is not semiprime.

Let $R = T_2(A_1)$ be the ring of the 2×2 upper triangular matrices with entries in the Weyl algebra A_1 (polynomials in x, y such that $xy - yx = 1$). Then R is not semiprime, but the commutator of scalar matrices generated by x and y is the identity matrix.

Finally, we give some well known examples of rings that have commutators nonzero divisors.

Example 21 *1. Any noncommutative ring without zero divisors, the matrix algebra over a division ring.*

2. Consider the 2×2 *matrix algebra* $\mathcal{M}_2(D)$ *over a domain* D *with* 1. *Let* E_{ij} *be the usual matrix units. Then the commutator* $[E_{12}, E_{21}] = E_{11} - E_{22}$ *is invertible.*

When $char(D) \neq 2$, *then in* $\mathcal{M}_3(D)$ *we have that* $[E_{12} + E_{23}, E_{21} - E_{32}] = E_{11} - 2E_{22} + E_{33}$ *is a nonzero divisor.*

Clearly variations of this will work for $\quad_n(D)$ *where* D *is a noncommutative domain (just consider* 2×2 *block diagonal matrices and a* 3×3 *block at the bottom if* n *is odd). This block matrix idea will also work for the ring of all (countably infinite) row and column finite matrices over a domain.*

In rings like $R = F[x, y]/(x^2)$*, the zero divisors must lie in* xR *or* Rx*, so* $[x, y]$ *is regular.*

Of course once one has suitable examples of prime rings, then direct sums give examples for semiprime rings.

In The Future. . .

(R. Awtar, '88): Let R be a non-commutative prime ring of characteristic 2, and let d be an additive mapping from R into itself satisfying $d(xy+yx)=d(x)y+xd(y)+d(y)x+yd(x)$ and $d(x^3)=x^2d(x)+xd(x)x+d(x)x^2$, for all $x,y\in R$. Then d is a derivation of R.

(Ashraf and Rehman, ’00): Let R be a 2-torsion free prime ring and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Is d is an additive mapping of R into itself satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then $d(uv) = ud(v) + vd(u)$ for all $u, v \in U$.

Now we want to address a question studied in one paper with Miguel Ferrero[a].

Some of these results are contained in the Ph.D. thesis of the lecturer[b].

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[a] *"Higher derivations of Semiprime Rings"*, Communications in Algebra 30(5) 2002, 2321-2333

[b] *"Derivações de Ordem Superior em Anéis Primos e Semiprimos"*, Ph.D. thesis, Universidade Federal do Rio Grande do Sul: Brasil (2000), p. 1-64

HD with Linear Relations

Throughout this part R is a semiprime ring, that is, $I \lhd R$ and $I^2 = 0 \Rightarrow I = 0$.

It is well known that if R is semiprime it has a (right or left) ring of quotients Q such that if $I \lhd_e R$ and $f\colon {}_R I \to {}_R R$, $\exists q \in Q$ such that $f(a) = aq$, for all $a \in I$.

Note that the center C of Q is a von Neumann regular commutative ring. If $I \lhd R$, then $I \bigoplus Ann(I)$ is essential in R. So the mapping $f\colon I \bigoplus Ann(I) \to R$, $f(a+b) = a$ defines an idempotent element $e_I \in Q$ with $e_I(a+b) = e_I a = a$, for all $a \in I$.

A derivation d: $R \to R$ is algebraic if there exist $a_1, \ldots, a_n \in R$, $a_n \neq 0$, such that $\sum_{i=0}^{n} a_i d^i(x) = 0$, for all $x \in R$.

Algebraic derivations have been studied first by V.K. Kharchenko[a] in a celebrated paper, then by A. Leroy and J. Matczuk[b] in prime rings and by A. Ouarit[c] in semiprime rings.

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[a] *"Differential Identities of Prime Rings"*, Algebra i Logika 17 (2) (1978), p. 220–238 = Algebra and Logic 1 (1978), p. 155–168

[b] *"Derivations et Automorphismes Algebriques d'Anneux Premiers"*, Communications in Algebra 13 (6) (1985), p. 1245–1266

[c] *"Dérivations et Automorphismes Algébriques d'Anneaux Semi-premiers*, C. R. Acad. Sci. Paris 314 (1) (1992), p. 241-244

It is well known that if R is semiprime and $d : R \to R$ is a derivation of R, then d can be uniquely extended to $d^\star$: $Q \to Q$ and $d^\star\mid_C$: $C \to C$.

In the above paper of A. Leroy and J. Matczuk the authors proved that d is R-algebraic (Q-algebraic), $d^\star$ is R-algebraic (Q-algebraic), $d^\star\mid_C$ is C-algebraic (Q-algebraic) are all equivalent conditions.

If $d : R \to R$ and $\exists a \in R$ such that $d(x) = xa - ax$, then d is said to be inner. Notation: $d = \delta_a$.

If d: $R \to R$, R prime, and there exists $q \in Q$ such that $d = \delta_q \mid_R$, then d is said to be X-inner (after Kharchenko).

Theorem 22 *(Kharchenko) Assume that R is prime and d is an algebraic derivation of R. If $char(R) = 0$ or $char(R) = p$ is greater than the degree of algebraicity of d, then d is X-inner.*

It is then natural to study relations $\sum_{i=0}^{n} r_i d_i = 0$, for a HD $D = (d_i)_{i \geq 0}$.

Our Results

Proposition 23 *Assume that R is semiprime and $D = (d_i)_{i \in \mathbb{N}}$ is a HD of R. Then there exists a unique extension of D to a HD $D^\star = (d_i^\star)_{i \in \mathbb{N}}$ of Q. Also $D^\star\,|_C$ is a HD of C.*

Definition 24 *Assume that $R \subseteq S$ or $S \subseteq R$. We say that D satisfies an S-linear relation (S-LR, for short) on R if there exist $a_0, a_1, \ldots, a_n \in S$, $a_n \neq 0$, such that $\sum_{i=0}^{n} a_i d_i(x) = 0$, for all $x \in R$.*

Theorem 25 *Assume that* R *is semiprime and* $D = (d_i)_{i \in \mathbb{N}}$ *is a HD of* R *which satisfies an* R*-LR of minimal lenght* n *on* R*. Then there exist* $I \lhd R$ *and* $q_0 = e_I, q_1, \ldots, q_{n-1} \in e_I Q$ *such that* $\mathop{}\limits_{i=1}^{n} q_{n-i} d_i(x) = 0$*, for all* $x \in R$*. Moreover, the relation* $\mathop{}\limits_{i=1}^{n} q_{n-i} d_i = 0$ *is minimal on* R*,* $e_I d_1 = \delta_{q_1}$ *and*

$e_I d_m = \delta_{q_m} - \mathop{}\limits_{i=1}^{m-1} q_i d_{m-i}$*, for all* $2 \leq m \leq n$*.*

Applications

First Application: If R is prime we can write the above relation: $d_1 = \delta_{q_1}$. Moreover,

$$\begin{bmatrix} \delta_{q_1} \\ \delta_{q_2} \\ \delta_{q_3} \\ \vdots \\ \delta_{q_m} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ q_1 & 1 & \dots & 0 & 0 \\ q_2 & q_1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ q_{m-1} & q_{m-2} & \dots & q_1 & 1 \end{bmatrix}}_{A(m,q)} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_m \end{bmatrix}.$$

This gives a generalization of Kharchenko's result:

$$\begin{aligned}
\delta_{q_1} &= d_1 \Rightarrow d_1 = \delta_{q1} \\
\delta_{q_2} &= q_1 d_1 + d_2 \Rightarrow d_2 = \delta_{q_2} - q_1 \delta_{q_1} \\
d_3 &= \delta_{q_3} - q_2 \delta_{q_1} - q_1(\delta_{q_2} - q_1 \delta_{q_1}) \\
&\vdots
\end{aligned}$$

It is clear that $A(m,q)$ is invertible for any $m \leq n$, and so the above relations can be written using matrices.

Second Application:

Theorem 26 *Assume that R is a semiprime ring and D is a HD of R. Then the following conditions are equivalent:*

1. *D satisfies an R-LR on R;*
2. *D satisfies a Q-LR on R;*
3. *$D^\star$ satisfies an R-LR on Q;*
4. *$D^\star$ satisfies a Q-LR on Q.*

Furthermore, if one of these equivalent conditions holds, then the minimal length of the relations is always the same.

In this case this is no more related with C-algebraicity. The next example gives a HD D such that satisfies an R-LR, but $D^\star$ does not satisfy a C-LR.

Example 27 *Let $R = \mathbb{K}\langle x, y\rangle$. Define $D = (d_i)_{i\in\mathbb{N}}$ by $d_0(r) = r$, $d_i(r) = (-1)^i(y^i r - y^{i-1} r y)$, $\forall r \in R$, $i \geq 1$.*

Then $d_2(r) + y d_1(r) = 0$, for every $r \in R$, and D satisfies an R-LR of length 2 on R.

Nevertheless D does not satisfy a C-LR on R, where C is the extended centroid of R.

Recall that $C = \mathbb{K}$. *If for* $c_i \in \mathbb{K}$, $0 \leq i \leq n-1$, *we have* $d_n + \mathop{}\limits_{i=0}^{n-1} c_i d_i = 0$, *then*

$$(-1)^n (y^n r - y^{n-1} r y) + \sum_{i=1}^{n-1} (-1)^i c_i (y^i r - y^{i-1} r y) + c_0 r = 0,$$

for every $r \in R$. *In particular, for* $r = x$ *we obtain*

$$(-1)^n y^n x = (-1)^{n+1} y^{n-1} x y - \sum_{i=1}^{n-1} (-1)^i c_i (y^i x - y^{i-1} x y) - c_0 x,$$

which is clearly a contradiction.

Some Open Problems on HD

HD and Partial Actions: the concept of a partial action of a group G on an abstract set X was introduced by R. Exel in 1998 as a family of partial bijections of X satisfying natural compatibility conditions.

It also can be defined as a partial homomorphism (partial representation) from G to the symmetric inverse semigroup $\mathcal{I}(X)$ of X, a rather natural concept studied in the theory of inverse semigroups.

Recently, in a pure algebraic context, partial representations and partial actions of groups on algebras have been studied by a lot of researchers which observed the relevance of partial actions in model theory, group presentations, and various aspects in the theory of semigroups.

Given a partial action of a group on an object it is natural to ask whether it is a restriction of a global action (globalization or enveloping action) defined on a bigger object. Globalizations of partial actions where first considered by F. Abadie in his PhD Thesis of 1999.

How could we insert HD in the partial action context?

HD and Hopf Algebras: the notion of action of a Hopf algebra in an algebra is enough general so that the ideas of action of a group in an algebra by automorphisms, or of action of a Lie algebra by derivations, or, still, to graduation by a finite group, for example, are dealt by an unified way.

This property of the Hopf algebras theory contributed for the appearence, in the 80', of a interest, by researchers in noncommutative ring theory, in the study of the action of Hopf algebras and in the search of generalizations of known results already concerning action of groups by automorphisms and of Lie algebras by derivations.

How could we insert HD in the Hopf algebras context?

I think we can consider the following structure:

$$R \longrightarrow k\langle x_1, x_2, \ldots, x_n\rangle,$$

$$\Delta(xy) = \qquad x_i \otimes x_j,$$

$$\epsilon(1) = 0.$$

Considering powers of one usual derivation, we would have $d^i d^j = d^{i+j}$.

Since $d_i d_j$ doesn't imply d_{i+j}, then R must have the algebra as a free ring, without relations.

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