A QUATERNION APPLICATION TO CONTROL ROTATION MOVEMENTS IN THE THREE DIMENSIONAL SPACE OF AN ARTICULATE MECHANICAL ARM TYPE ROBOT BUILT FROM LOW COST MATERIALS AS A SUPPORTING TOOL FOR TEACHING AT THE UNDERGRADUATE LEVEL

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Abstract *— Quaternions have developed a widespread use in computer graphics and robotics research because they can be used to control rotations in the three dimensional space. Our main proposal is using an articulate mechanical arm type robot, built from low cost materials, as a supporting tool for teaching-learning of Mathematics at the undergraduate level. We show an unusual representation form of the quaternions, simplifying the formulas of the control system for the direct and inverse kinematic models. This modeling proved to be simple and allows to the student a more dynamic and practical learning of the concepts developed in Linear Algebra, Numerical Methods, Computer Graphics and Programming Languages.*

Index Terms — Quaternions, Engineering Education, Mathematical Modeling, Robotics, Computer Graphics.

1. INTRODUCTION

In continuity to our research developed in [5], we shall model mathematically the movements of our articulated mechanical arm type robot, now in the 3D space case. An interesting way to describe the rotation movements in 3D is by using quaternions, which can be proven by computation engineers who work with Computer Graphics.

In this article, we show how this quaternions approach can contribute as a supporting tool for teaching at the undergraduate level, specifically in Linear Algebra, Analytic Geometry, Calculus III, Computer Graphics, Numerical Methods and Programming Languages, getting students to learn basic programming and robotics by using educations models. This is the main pourpose of technological tools: to be the link between theory and practice.

UNIVATES has set its engineering courses recently and supports the adoption of technological tools to help overcome students' limitations both in study time and in previous learning experiences, as well as their complete lack of self-learning abilities. This makes them feel closer to work situations, motivating them learn more and on their own ([25]).

To fill out this gap, two robots were built in the development of a research that had the students' grant holders of scientific initiation participation, besides having the voluntary students' participation joined to the project in several moments of the research. We trust that an articulate mechanical arm type robot opens possibility to explore a wide set of contents in undergraduate courses.

As we have described in [3], [4], [5], and [6], the robots were built with motors of continuous current of constant magnetic field acquired in junk-dealer trade. The structure was made of wood. The axes of the joints were built on bicycle cubes, and the controlling plates were also developed in UNIVATES using recycled electronic components. Each robot has a fixed base, a rotative vertical support, an upper arm (l1), a forearm (l2), a rotating pulse, and a magnetic claw in the extremity. Therefore, with 4 degrees of freedom (DOF, for short) ([5]). The revolution joints of the forearm and of the upper arm lie in the same plane and the robot's claw holds objects through electromagnetism (Figure 1).

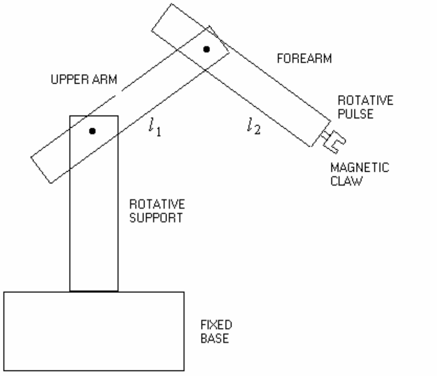

FIGURE. 1

ARTICULATE MECHANICAL ARM TYPE ROBOT.

This article is organized in the following way: in the Section 2 it is made a description of the ring of quaternions; in the Section 3 we present an unusual representation of quaternions; in the Section 4 we describe rotations using quaternions; in the Section 5 we give a matrix representation for quaternions; in the Section 6 we present some simple applications. Finally, in the Section 7 we present the conclusions and future works.

We recall that this paper on quaternions was written as a resource for regular and scientific initiation students, and professors. We didn't discover anything original about them.

2. THE RING OF QUATERNIONS

Following [16], Sir W.R. Hamilton (1805-1865) gave the definitive foundation for the complex numbers as ordered pairs of real numbers, such as it is developed currently. Hamilton was also a notable physicist and perceived clearly the implications of his discovery: he had developed an algebra that allowed to work with vectors in the plane. This led him to consider a problem whose solution would be fundamental for the physics of his time: to develop an algebra of 3-tuples that would give an appropriate language to work with 3D vectors, and thus to develop the dynamics.

Roughly speaking, quaternions are an extension of complex numbers. Instead of just i, we have three different numbers that are all square roots of -1 labelled i, j, and k.

The quaternion number system had decisive importance in at least two contexts. On one hand, they had originated vectorial calculus. On the other hand, the discovery of quaternions had a decisive role in the development of algebra: they had the virtue to show that the fundamental laws suggested by the systems known until 1843, were not something to being assumed always, *a priori*. The set of quaternions is the first known example where the order of the factors modifies the product, that is, the first noncommutative algebra. So they had also shown clearly the possibility to still extend more the set of known algebras ([10]).

To construct the quaternion ring, let us consider the set of all formal linear combinations of i, j, and k with real coefficients:

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$$H=\{a+bi+cj+dk\,|\,a,b,c,d\in\mathbb{R}\}.$$

Addition of two quaternions is very straighforward: we just add the coefficients. That is, if $q_1=a_1+b_1 i+c_1 j+d_1 k$ and $q_2=a_2+b_2 i+c_2 j+d_2 k$, then the sum is

$$q_1+q_2=(a_1+a_2)+(b_1+b_2)i+(c_1+c_2)j+(d_1+d_2)k.$$

Multiplication is somewhat more complicated. The product is defined by distributivity, assuming that the coefficients commute with the symbols *i, j,* and *k*, and that these multiply to each other according to the following rules:

$$i^2=j^2=k^2=-1\;, ij=-ji=k\;, jk=-kj=i\;, ki=-ik=j\,.$$

So the product of q_1 and q_2 is

$$q_1*q_2=(a_1a_2-b_1b_2-c_1c_2-d_1d_2)+(a_1b_2+a_2b_1+c_1d_2-d_1c_s)i$$
$$+(a_1c_2+a_2c_1+d_1b_2-b_1d_2)j+(a_1d_2+a_2d_1+b_1c_2-c_1b_2)k\,.$$

It is easy to see that, with the operations defined above, the set H is a noncommutative associative ring. Furthermore, H is a division ring ([16]).

3. AN UNUSUAL REPRESENTATION OF QUATERNIONS

We can represent a quaternion in several ways ([14], [17]):

- as a linear combination of 1, *i, j,* and *k*;
- as a vector of the four coefficients in this linear combination;
- or as a scalar for the coefficients of 1 and a vector for the coefficients of the imaginary terms.

In this last form, the quaternion $q=a+bi+cj+dk$ is written as $(a,\vec{v})$, where $\vec{v}$ is the 3D vector $\vec{v}=(b,c,d)$. This dramatically simplifies the operator formulas of addition and multiplication. We can rewrite the addition formula for two quaternions $q_1=(a_1,\vec{v}_1)$ and $q_2=(a_2,\vec{v}_2)$ as $q_1+q_2=(a_1+a_2,\vec{v}_1+\vec{v}_2)$ and the product formula as

$$q_1*q_2=(a_1a_2-\vec{v}_1\cdot\vec{v}_2,a_1\vec{v}_2+a_2\vec{v}_1+\vec{v}_1\times\vec{v}_2)\,,$$

using the standard vector products (inner product and cross product) studied in Linear Algebra and Analytic Geometry.

Recall that quaternion numbers of the form $(a,\vec{0})$, where $\vec{0}=(0,0,0)$, can be associated with the real numbers, while the ones of the form $(a,\vec{v})$, where $\vec{v}=(b,0,0)$, can be associated with the complex numbers.

The *i, j, k* representation for quaternions can be shown to be identical with these formulas, however this results in even worse complexity for the division and inverse formulas. So the $(a,\vec{v})$ representation, although unusual for mathematicians, have become the preferred form by engineers that work with Computer Graphics and Robotics, as we will see with more details later.

With this new representation, it is straightforward to develop a complete set of properties of the division ring H. Given a quaternion number $q=(a,\vec{v})$ as above, we can compute easily the additive inverse $-q=(-a,-\vec{v})$, the conjugate $\bar{q}=(a,-\vec{v})$, and mainly the inverse

$$q^{-1}=\left(\frac{a}{a^2+|\vec{v}|^2},-\frac{\vec{v}}{a^2+|\vec{v}|^2}\right).$$

Quaternions of the form $(a,\vec{0})$ are normally denoted in their real number form – as a. If *c* is a scalar, then we will consider $c*q=(c,\vec{0})*(a,\vec{v})=(ca,c\vec{v})$. It also allows us to simplify some expressions. For example, que inverse q^{-1} is given by $q^{-1}=\frac{(a,-\vec{v})}{a^2+|\vec{v}|^2}$, where $|\vec{v}|$ is the lenght of $\vec{v}$, namelly, $|\vec{v}|=\sqrt{b^2+c^2+d^2}$. Finally, we define the length of *q* to be $|q|=\sqrt{a^2+|\vec{v}|^2}$.

4. REPRESENTATING ROTATIONS WITH QUATERNIONS

We can use quaternion multiplication to perform a rotation about an arbitrary unit axis $\vec{u}$ by the angle θ. A unit vector denotes here a vector with length 1. The quaternion that computes this rotation is $q=(a,\vec{v})$, where $a=\cos\left(\frac{\theta}{2}\right)$ and $\vec{v}=\vec{u}\sin\left(\frac{\theta}{2}\right)$. Note that *q* is a unit quaternion, therefore q^{-1} is the conjugate of *q*.

We will represent a point P in the 3D space by the quaternion $p=(0,\vec{P})$. The desired rotation of that point is given by this formula: $P_{rotated}=q*p*q^{-1}$.

Do quaternions really perform 3D rotations?

Following K. Shoemake, quaternion multiplications preserves norms, that is, $N(p*q)=N(p)N(q)$. This also implies $N(q^{-1})=N(q)^{-1}$. Hence $N(q*p*q^{-1})=N(p)$. This proves that the map $p\rightarrow q*p*q^{-1}$ is an orthogonal transform in 4D. On the other side, scalar multiplication commutes, so $q*a*q^{-1}=a$. Thus $p\rightarrow q*p*q^{-1}$ is an orthogonal transform of the 3D vector of *p*. There is a continuous path from the identity to every possible action. This excludes reflections, so the map $p\rightarrow q*p*q^{-1}$ is a 3D rotation.

Suppose now that we want to perform two rotations on the robot's claw. This may come up in a manipulation interface where each movement adds another rotation to the current claw position. Let q_1 and q_2 be two unit quaternions representating two rotations. We want to perform q_1 first and then q_2. The composite rotation is then represent by

$$q_2*q_{1rotated}=(q_2*q_1)*p*(q_2*q_1)^{-1}.$$

Following [11], the representation of a rotation as a quaternion (4 numbers) is more compact than the representation as an orthogonal matrix (9 numbers). Also, for a given axis and angle, one can easily construct the corresponding quaternion, and conversely, for a given quaternion one can easily read off the axis and the angle. Both of these are much harder with matrices or Euler angles ([23], [24]).

In computer games and other applications, one is often interested in "smooth rotations", meaning that the scene should slowly rotate and not in a single step. This can be accomplished by choosing a curve in the quaternions, with one endpoint being the identity transformation 1 and the other being the intended total rotation. This is more problematic with other representations of rotations.

When composing several rotations on a computer, rounding errors necessarily accumulate. A quaternion that's slightly off still represents a rotation - a matrix that's slightly off need not be orthogonal anymore and therefore need not represent a rotation at all. It is hard to turn such a matrix back into a proper orthogonal one.

The orthogonal matrix corresponding to a rotation by the unit quaternion $q=a+bi+cj+dk$ is given by:

$$\begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2ad+2bc & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

5. REPRESENTATING QUATERNIONS BY MATRICES

We can also representate quaternions by matrices ([10]): there are at least two ways of representing quaternions as matrices, in such a way that quaternion addition and multiplication correspond to matrix addition and matrix multiplication (i.e., quaternion-matrix homomorphisms).

One is to use 2×2 complex number matrices, and the other is to use 4×4 real number matrices.

In the first way, the quaternion $q=a+bi+cj+dk$ is represented as: $\begin{pmatrix} a-di & -b+ci \\ b+ci & a+di \end{pmatrix}$. This representation has several nice properties:

- All complex numbers correspond to matrices with only real entries;
- The square of the absolute value of a quaternion is the same as the determinant of the corresponding matrix;
- The conjugate of a quaternion corresponds to the conjugate transpose of the matrix;
- Restricted to unit quaternions, this representation provides the group isomorphism between the 3D-sphere S^3 and *SU(2)*. The latter group is important in quantum mechanics when dealing with spin.

In the second way, the quaternion q as above is represented as:

$$\begin{pmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{pmatrix}.$$

In this representation, the conjugate of a quaternion corresponds to the transpose of the matrix.

6. APPLICATIONS

Quaternions are often used in Computer Graphics (and associated Geometric Analysis) to represent rotations and orientations of objects in 3D space. They are smaller than other representations such as matrices, and operations on them such as composition can be computed more efficiently. Quaternions also see use in Control Theory, Signal Processing, Attitude Control, Pphysics, and Orbital Mechanics, mainly for representing rotations/orientations in three dimensions. For example, it is common for spacecraft attitude-control systems to be commanded in terms of quaternions, which are also used to telemeter their current attitude. The rationale is that combining many quaternion transformations is more numerically stable than combining many matrix transformations.

This Section is dedicated to present a list of some simple applications of quaternions in Computer Graphics.

- Using quaternions to perform linear interpolation between matrices ([13], Q61):

For many animation applications, it is necessary to interpolate between two rotation positions of a given object. These positions may have been specified using keyframe animation or inverse kinematics.

- Using quaternions to perform cubic interpolation between matrices ([13], Q62):

For some applications, it may not be convenient or possible to use linear interpolation for animation purposes. In this case, cubic interpolation is another alternative. In order to use cubic interpolation, at least four rotation matrices must be known. Each of these is then converted into a set of spherical rotations via quaternions and spherical rotation angles (ie. longitude, latitude and rotation angle). These are then multiplied with the base matrix for a Cardinal spline curve. This interpolation matrix can then be used to determine the intermediate spherical rotation angles. Once the interpolated coordinates are known (latitude, longitude and rotation angle), the interpolated rotation matrix can then be generated through the conversion to quaternions.

- Using quaternions to rotate a vector ([13], Q63):

A rather elegant way to rotate a vector using a quaternion directly is the following: $\vec{v}_{rot}=q_{rot}*\vec{v}*q_{rot}^{-1}$, where q_{rot} is the rotation quaternion. This can easily be realised and is most likely faster then the transformation using a rotation matrix.

More applications can be obtained in [7], [8], [9], and [12].

7. CONCLUSIONS AND FUTURE ACTIVITIES

With these representations, we can describe the kinematics models of movements of our robot. In fact, recalling that the kinematics direct model consists of the description of the position and of the orientations of the claw in function of the variables of the links, while the inverse ones consists of the determination of the variables starting from the position and the orientation of the claw, we obtained a considerable progress, since in [5] we had just described these movements in the bidimensional case.

The robots' use in our classes has showed for the undergraduate engineering students more clearly the applications of some indigestible topics of mathematics such as inner product, vector product, linear systems and complex numbers. Besides, we have been able to relate first year disciplines with more advanced ones, such as Numerical Methods, Computer Graphics, and Programming Languages, allowing to the student an early contact with the technology. However, the use of technology doesn't invalidate the regular theorical classes, especially in the mathematical ones. Such traditional theoretical classes are very important too, and we should not imagine that the use of the technology should simply substitute them. The theory is indispensable so that the student gets to relate the contents with the practical applications.

As a future work we will study systems of nonlinear equations of several variables, using Gröbner Bases to determine the position of the robot's claw in any point of your path, as well as the dynamic models of movements.

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