ON FREE ACTIONS AND DEPENDENT ELEMENTS

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International Conference on Algebra and its Applications

20-22 February, 2010

Aligarh Muslim University, India

http://www.amu.ac.in/conference/maths.htm

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This work is based on Chapter 7 of the PhD thesis of the first named author.

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1. ON DEPENDENT ELEMENTS

In this section we study the dependent elements of a map $f: R \rightarrow R$, where R is a ring.

The dependent elements are the elements $a \epsilon R$ which satisfy the relation $f(x)a=ax$, for all $x \epsilon R$.

If there exists no nonzero element a which satisfies that relation, we cal the map f a free action.

Free action maps also will be studied in this section.

The main idea is to study the functions satisfying $f(x)a=ax$, for all $x \epsilon R$ and some fixed element $a \epsilon R$.

Example 1

If R has an identity element and a is invertible, we can characterize all functions with dependent elements a.

The function $f(x)=axa^{-1}$ has a for dependent element, because $f(x)a=axa^{-1}a=ax$.

Furthermore, if $f(x)a=ax$ for all $x \in R$, then $f(x)=axa^{-1}$.

Example 2

If $Ra=R$ then given any $x \in R$, there exists some $z \in R$ such that $za=ax$.

Let $f(x)$ pick one such $z \in R$. Then $f(x)a=za=ax$.

Notice that f is not necessarily additive nor multiplicative.

Theorem 1

Given $f: R\rightarrow R$ *with dependent element* a, *then* $I=\{x\in R \mid ax=0\}$ *is an ideal.*

Remark 1

If f is not 1-1, then $f(x)=f(y)$ implies $x-y\in I$, for all $x,y\in R$.

Remark 2

$I=\{x \mid ax=0\}=\{x \mid f(x)a=0\}$.

Remark 3

The kernel of f is contained in I.

Theorem 2

The set $H=\{x|\ a^n x=0$ *for some* $n\}$ *is an ideal.*

We now look at annihilators of a on the other side.
There are a lot of them.
But they do not form an ideal.

Theorem 3

Given $f: R\rightarrow R$ *with dependent element* a*, then* $\{f(xy)-f(x)f(y)\}a=0$*, and* $\{f(x+y)-f(x)-f(y)\}a=0$.

Definition 1

$W=\{w\epsilon R|\ wa=0\}$.

Theorem 4

W is a left ideal.

Theorem 5

WRa is a two sided ideal.

Theorem 6

WRa^n is an ideal, for any natural number n.

Remark 4

There is a decreasing chain of two-sided ideals

$$WR \supseteq WRa \supseteq WRa^2 \supseteq \ldots \supseteq WRa^n \supseteq WRa^{n+1} \supseteq \ldots$$

Theorem 7

Given f: $R \to R$ be a homomorphism with dependent element a, then $f(W)a^2=0$, where W is defined in Definition 1.

Theorem 8

$f^n(W)a^{n+1}=0$, *for any natural n.*

Remark 5

If $aR=R$, then $WR=0$ and W is a trivial ideal.

Remark 6

If $f(R)=R$, then $WR=Wf(R)$ is contained in W and W is an ideal.

Remark 7

f(I) is contained in W.

Remark 8

If we define f´(x)=f(x)+w(x) for any choice of w(x) in W, then f´also works for the dependent element a, that is f´(x)a=(f(x)+w(x))a=f(x)a=ax.

Remark 9

Let W, f be as defined above. Then

$$Q = \{x \in R \mid Wx \subset W\}$$

is a subring.

Moreover, {a}, f(R) and W are contained in Q.

Remark 10

The subring Q contains a.
The restriction of f maps Q into Q.
And, of course, $f(q)a=aq$, for all $q \in Q$.

In Q, W is a two sided ideal.
It is not obvious that we can define f on the equivalence classes of Q/W because we do not know that W is invariant.
Yet is should be possible because modifying f by elements in W always yields a valid function with dependent element a.

Pick any representative from each equivalence class. Then: $f'(x+W)(a+W)=(f(x+w)+W)(a+W)=$ $=f(x+w)a+W=(f(x)+f(w))a+W=f(x)a+W$, because $f(W)a$ is contained in W.

Theorem 9

$\Omega=\{x\epsilon R|\ xR$ is contained in $W\}$ is an ideal of R.

Theorem 10

$f(R)a=aR$ is a right ideal.

Theorem 11

Ra is a two sided ideal.

Suppose that a and b are both dependent elements for f.

Thus f(x)(a+b)=f(x)a+f(x)b=ax+bx=(a+b)x.

So the sum of two dependent elements is also a dependent element.

Since f(f(x))ab=af(x)b=abx, then the product fo two dependent elements is a dependent element for f(f(x)).

Consequently, $f^n(x)a^n=a^nx$.

This yields the following descending chain of ideals:

$$Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \ldots \supseteq Ra^n \supseteq Ra^{n+1} \supseteq \ldots$$

Theorem 12

$(Ra)^n$ is contained in Ra^n.

Theorem 13

If a is nilpotent, then Ra is a nilpotent ideal.
In a semiprime ring, if a is nilpotent, then $a=0$.

Remark 11

Look at the subring generated by a and the function f.

One can write the basis as products of the form $a_k b_1 b_2 \ldots b_n$, where the b_is are defined inductively.

The original B_is are $B_1=\{f(a),f(a_2),\ldots,f(a_n),\ldots\}$ and by taking all finite products.

Now close B_1 by taking all images by f.

This is B_2.

Now close B_2 by taking products and then all images by f, etc.

When two monomials are multiplied, the results are simplified by bringing all of the singleton a´s to the left end.

Remark 12

Left multiplication by a is 1-1 but not onto.

Remark 13

Right multiplication by a´s collapses things.
If one multiplies by a large enough power of a, the whole term collapses to a single power of a.

Remark 14

If R is k-torsion-free, where k is a fixed number, and f satisfies some dependence relation which is homogeneous and such that the coefficients do not add up to zero, then a generates a trivial ideal.

Assume $f(x)a=ax$.
Since $pf(x)p^{-1}pa=pf(x)a=pax$, we find that $g(x)=pf(x)p^{-1}$ is a function that has dependent element pa.

If R is a ring of $n\times n$ matrices, by the above statement we may assume that a is in row canonical form since $g(x)pa=pax$, for all $x\epsilon R$.

If a is invertible then $f(x)=axa^{-1}$ and we know g exactly.
If a is not invertible, then the row canonical form of a is not the identity matrix.

Suppose that the i^{th} column of a does not contain a stair step one.
The product aE_{ii} has the same i^{th} column of a and all other entries are zero.
This matrix aE_{ii} can not be obtained by any left multiplication.
Thus all elements aE_{ii} must be zero and the i^{th} column of a all zeros.

But we can put this column of zeros anywhere in the matrix by a right multiplication.

We cannot do that by a left multiplication.
Therefore a can have no nozero entries.
Therefore a=0.
So for the ring of nxn matrices, any nonzero function is a free action.

Assume that A is a prime ring and $f(x)a=ax$.
Then Ra is na ideal.

If $aw=0$, then $(RaR)(RwR)$ is contained in $RawR=0$, and so in the prime case, $w=0$. (*)

If $f(p)=0$ then $f(p)a=ap=0$ and so $p=0$.

If $f(p)=f(q)$ then $0=(f(p)-f(q))a=a(p-q)$ so $p=q$.
Therefore in a prime ring f is 1-1.

If $f(p)a=0$ the $ap=0$ and so $p=0$.

If $W=\{x\in R|\ xa=0\}$ then W is a left ideal, $RWf(R)$ is contained in W and $f(R)\cap W=0$, because $f(x)\in W$ means $f(x)a=0$ then $0=f(x)a=ax=0$.
Since $ax=0$, then from (*), $x=0$.

Since $x=0$ then $f(x)=0$.
Therefore $f(R)\cap L(a)=0$, where $L(a)$ is the left annihilator of a in R.

$f(p+q)a=a(p+q)=ap+aq=f(p)a+f(q)a$.
So $(f(p+q)-f(p)-f(q))a=0$, so “$f(x)Ra$” is additive.

$f(pq)a=apq=f(p)aq=f(p)f(q)a$.
So $(f(pq)-f(p)f(q))a=0$, so $f(x)$ is sort of multiplicative.

Theorem 14

If R is a ring and aR is contained in Ra for some $a \epsilon R$*, then there is a dependent function with dependent element a.*

Corollary 15

If aR is contained in Sa for some subset S of R, then g(R) is contained in S.

Corollary 16

Let $f(x)a=ax$ *be a dependent function with its dependent element a.*

Then $S= \text{Range } f=\{f(x) \mid x \epsilon R\}$ *gives that aR is contained in Sa, which is a characterization of all dependent functions with their dependent element.*

The following construction seems to be way too complicated.

Take any ring R and any element a.

Let $Q=<(x_1,x_2,\ldots,x_i,\ldots)>$, where $ax_i=x_{i+1}a$, for every i.

Q is closed under coordinatewise addition and coordinatewise multiplication.

If $ay_i=y_{i+1}a$ for all i and $az_i=z_{i+1}a$ for all i, then $a(y_i+z_i)=(y_{i+1}+z_{i+1})a$ for all i.

So $a(y_iz_i)=(ay_i)z_i=y_{i+1}az_i=y_{i+1}z_{i+1}a$ for all i.

Let H_i be all the coordinates appearing in the i^{th} slot.

Then $H_{i+1}a=aH_i$.

Also notice that one can start anywhere in the sequence, so H_n is contained in H_1 for all n.
Then the set of all first coordinates $H=H_1$ is a subring which satisfies aH is contained in Ha, and conversely, if f(x)a=ax then the sequence (x,f(x),f(f(x)),…) has the above property.
$f^{i+1}(x)a=af^{i}(x)$, or
$H=\{x\text{: for each } n\text{, there exists } y_n \text{ s.t. } a^n x=y_n a^n\}$.

H is closed under addition and multiplication so is a subring.

Essentially, we are saying that each element in the ring has an infinite number of predessors.
They are etc … f(f(f(x))),f(f(x)),f(x),x, i.e., $y_n=f^n(x)$.

Theorem 17

If R is a ring with identity and a is an element in R with left inverse b, that is, $ba=1$*, then* $g(x)=axb$ *is a dependent function with dependent element a.*

Definition 2

If R is a ring with two dependent functions f an g with the same dependent element a, we say that f and g are equivalent if $f(x)a=g(x)a$ for all $x\in R$.

It might not be true that $f(x)=g(x)$ for all $x\in R$, but at least $(f(x)-g(x))a=0$.

If f is a dependent function with dependent element a, let $W=\{w| \ wa=0\}$, then any dependent function f can be modified by adding elements of W to it.

It will still be a dependent function with dependent element a, i.e., if $u: R\rightarrow W$, then $g(x)=f(x)+u(x)$ is a dependent function with dependent element a,

$g(x)a=(f(x)+u(x))a=f(x)a=ax$.

We classify all dependent functions on simple rings with identity element.

We show that they are all equivalent to some $g(x)=axb$ for some elements a an b.

If $a=0$, then choose b to be zero also.

When a is not zero then b is the left inverse of a.



Theorem 18

Assume that R is a simple ring with identity element. If there exists a function $f: R \rightarrow R$ such that $f(x)a=ax$, for all $x \in R$, then f is equivalent to $g(x)=axb$ for some $b \in R$.

We can define a dependent map for an element a if a has a “partial inverse”. That is, if there exists an element b such that ba acts like an identity on a, i.e., if $(ba)a=a$.

Theorem 19

If $a,b \in R$ such that $Ra=R$ and $ba^2=a$, then R has a dependent map with dependent element a.

Theorem 20

If R is a simple ring not necessarily with id and f is a dependent function with dependent element $a \neq 0$, then $Ra=R$ and there exists $b \in R$ s.t. $ba^2=a$ and the function g defined by $g(x)=ayba$, where $x=ya$, is equivalent to f.

2. ON THE RELATION f(x)a=bx

In this section we study some kinds of mappings $f: R \to R$ which satisfy the condition $f(x)a=bx$, for all $x \in R$ and some fixed elements $a,b \in R$.

This property is a generalization of the definition of one dependent element.

Definition 1

Let R be a ring. Let f: R→R be a mapping.
We call that a is associated with b in R if f(x)a=bx, for all xϵR.

We shall denote by
A(f)={(a,b)ϵRxR: f(x)a=bx, for all xϵR}, the set of associated pairs of f.

Theorem 1

A(f) is an additive subgroup of RxR.

Theorem 2

If (a,b)ϵA(f) then $(a^n,b^n)\epsilon A(f^n)$ *for every natural n.*

Definition 2

A derivation D of a ring R is an additive mapping $D: R \rightarrow R$ which satisfies $D(xy)=D(x)y+xD(y)$ for every $x,y \in R$.

Definition 3

A generalized derivation is a mapping $G: R \rightarrow R$ which is additive and satisfies $G(xy)=G(x)y+xD(y)$ for every $x,y \in R$, and some derivation D in R.

Theorem 3

Let R be a semiprime ring and let $g: R \rightarrow R$ be a generalized derivation related with a derivation $d: R \rightarrow R$. If $(a,b) \in A(g)$, then $ab=ba$.

Theorem 4

Let R be a 2-torsion-free prime ring and let d, g and h be nonzero derivations of R.

Suppose f be the mapping $x \longrightarrow d(g(x))+h(x)$.

If $(a,b) \epsilon A(f)$, then $ba=0$.

Theorem 5

Let R be a prime ring and let T be a nonzero left centralizer of R.

If $(a,b) \epsilon A(T)$, then $a \epsilon Z(R)$, the center of R.

Definition 4

A left centralizer T of a ring R is a mapping $T: R\to R$ which is additive and satisfies $T(xy)=T(x)y$ for every $x,y\epsilon R$.

Theorem 6

Let R be a semiprime ring and let T be a nonzero left centralizer of R. If $(a,b)\epsilon A(T)$, then $ab=ba$.

Theorem 7

Let R be a prime ring and let g be a generalized derivation of R related with a derivation d of R. Assume that $\theta: R\to R$ is such that $\theta(x)=[g(x),x]$. If $(a,b)\epsilon A(\theta)$, then $ba=0$.

Definition 5

Let R be a ring and let α be a nonzero mapping of R. An additive mapping $D: R \rightarrow R$ is called an α-derivation if $D(xy)=D(x)y+\alpha(x)D(y)$ holds for all $x,y \epsilon R$.

Theorem 8

Let R be a prime ring and let d be a nonzero α-derivation, where α is na homomorphism of R. Assume that $\theta: R \rightarrow R$ is such that $\theta(x)=d(x)+\alpha(x)$. If $(a,b) \epsilon A(\theta)$, then $(a,b) \epsilon A(\alpha)$ or $ab=ba$.

Theorem 9

Let R be a semiprime ring and let f and T be two left centralizers of R.
Assume that $\theta: R\rightarrow R$ is such that $\theta(x)=(f\circ T)(x)$.
If $(a,b)\epsilon A(\theta)$, then $ab=ba$.

Corollary 10

Let R be a prime ring and let f and T be two left centralizers of R.
Assume that $\theta: R\rightarrow R$ is such that $\theta(x)=(f\circ T)(x)$.
If $(a,b)\epsilon A(\theta)$ then $b=0$ or $a\epsilon Z(R)$.

One detail report on derivations can be found in

Ashraf, M.; Ali, A. and Haetinger, C., *On derivations in rings and their applications*, The Alligarh Bull. of Math. **25**(2) (2006) 79-107.

I think that´s my say!

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