# REPRESENTATION THEORY OF SEMISIMPLE HOPF ALGEBRAS

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## 1. Introduction

Let $H$ be a finite-dimensional semisimple Hopf algebra over a field $k$. We usually assume that $k$ is algebraically closed of characteristic zero. In these notes we discuss the representation theory of such Hopf algebras, analogous to classical results about the representation theory of finite groups. We give fairly complete proofs of many of the results, since a number of the original proofs have been simplified, particularly by H.-J. Schneider and M. Lorenz.

We begin with a more general discussion of finite-dimensional Hopf algebras and their properties, and prove the semisimplicity criterion in terms of the antipode proved by Larson and Radford. We then specialize to the semisimple case, proving the orthogonality of characters and what is called the "Class Equation" for Hopf algebras. We then give a number of applications of the class equation, related to the classification of Hopf algebras and the dimensions of irreducible modules. We also discuss the Grothendieck ring $K_0(H)$ and its relation to twistings of Hopf algebras. Finally we consider the Schur indicator of an irreducible representation.

Throughout $H$ is a finite-dimensional Hopf algebra over $k$, with multiplication $m : H \otimes H \to H$, unit $u : k \to H$ (given by $u(1_k) = 1_H$), comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$, and antipode $S : H \to H$. We abbreviate $\Delta(h)$ using the so-called "sigma notation"; that is, $\Delta h = \sum h_1 \otimes h_2$. For example, let $H = kG$, the group algebra of the finite group $G$. $H$ becomes an algebra using the group operation, and a Hopf algebra by defining $\Delta x = x \otimes x$, $\varepsilon(x) = 1$, and $S(x) = x^{-1}$, for any $x \in G$. These maps are extended linearly to $kG$.

Since $H$ is finite-dimensional, its $k$-linear dual $H^*$ is also a Hopf algebra, with structure maps given by appropriate transposes of those of $H$. That is, $m_{H^*} = (\Delta_H)^*$, $u_{H^*} = (\varepsilon_H)^*$, $\Delta_{H^*} = (m_H)^*$, $\varepsilon_{H^*} = (u_H)^*$, and $S_{H^*} = (S_H)^*$. Thus multiplication in $H^*$ is given by *convolution*, that is $(f * g)(h) = m_{H^*}(f \otimes g)(h) = (f \otimes g)(\Delta h) = \sum f(h_1)g(h_2)$, for all $f, g \in H^*$ and $h \in H$. Similarly comultiplication in $H^*$ is given by $\Delta f(h \otimes l) = (m_H)^* f(h \otimes l) = f(hl)$, for all $f \in H^*$ and $h, l \in H$. In particular $(kG)^*$ is also a Hopf

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algebra: convolution becomes usual pointwise multiplication of functions, and the comultiplication can be given more explicity in terms of the dual basis $\{p_x | x \in G\}$ of $(kG)^*$: that is, $\Delta p_x = \sum_{y \in G} p_y \otimes p_{y^{-1}x}$.

There are several other Hopf algebras associated to $H$. $H^{op}$ denotes the Hopf algebra with opposite multiplication to $H$ but the same comultiplication, and $H^{cop}$ is the Hopf algebra with the same multiplication as $H$, but opposite comultiplication (equivalently, $H^*$ has opposite multiplication). We note that the antipode of both $H^{op}$ and $H^{cop}$ is $S_H^{-1}$, since the defining property of $S_H$ is that $S * id = id * S = \varepsilon$, where $*$ denotes the convolution product in $\mathrm{End}_k(H)$.

In general we will say a Hopf algebra is *non-trivial* if it is not isomorphic to $kG$ or $(kG)^*$ for any $G$. In any Hopf algebra, $G(H) = \{0 \neq h \in H | \Delta h = h \otimes h\}$ is the set of *group-like* elements in $H$; $kG(H)$ is a Hopf subalgebra of $H$ isomorphic to the group algebra of $G(H)$. Knowing $G(H)$, or even whether it is non-trivial, is extremely useful.

A fundamental property of the representation theory of Hopf algebras is that, similarly to the case of groups, the tensor product of $H$-modules is again an $H$-module. That is, let $\mathcal{C} = {}_H\mathcal{M}$ denote the category of left $H$-modules. If $V, W \in \mathcal{C}$, then $V \otimes W \in \mathcal{C}$, via $h \cdot (v \otimes w) := \Delta(h) \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$, for all $h \in H, v \in V, w \in W$. This tensor product is associative and has an "identity" $I = V_0$, where $V_0 = kv_0$ is the *trivial* module, that is $h \cdot v_0 = \varepsilon(h)v_0$. Thus in fact $\mathcal{C}$ is a *monoidal* category. Moreover for each $V \in \mathcal{C}$, the dual module $V^* \in \mathcal{C}$ also, via $(h \cdot f)(v) := f(Sh \cdot v)$ for all $h \in H, f \in H^*$ and $v \in V$; this will imply that the category $\mathcal{C}$ is *rigid*; see [Ksl].

More generally, for $V, W \in \mathcal{C}$, also $Hom_k(V, W) \in \mathcal{C}$ via $(h \cdot f)(v) := \sum h_1 \cdot (f(Sh_2 \cdot v)) \in W$; $V^*$ is the case when $W = V_0$.

Conversely if $A$ is a $k$-algebra and ${}_A\mathcal{M}$ is a rigid monoidal category, then $A$ can be given the structure of a Hopf algebra. Thus these elementary properties of the tensor product of $H$-modules can be considered as the defining properties of a Hopf algebra.

Note that it would be very useful to know that for any finite-dimensional $V \in \mathcal{C}$, $V^{**} \cong V$. This will happen if $S^2 = id$, or more generally if $S^2$ is an inner automorphism of $H$. The order of $S^2$ is considered Section 2.

We record here a few elementary facts about representations and characters. If $V$ is a left $H$-module, we write $\phi_V : H \to End(V)$ for the representation of $H$ on $V$. When $V$ is finite-dimensional, we let $\chi_V$ denote the character of the representation; that is, $\chi_V(h)$ is the trace of $\phi_V(h)$, for any $h \in H$. Note that $\chi_V \in H^*$. The following lemma is an easy exercise.

**Lemma 1.1.** *Let $V, W$ be left $H$-modules. Then as left $H$- modules,*

*(1) $(V \otimes W)^* \cong W^* \otimes V^*$,*

*(2) $V \otimes W^* \cong Hom(W, V)$, and*

*(3) $Hom_H(V, W) \cong Hom(V, W)^H$, where for any $H$-module $V$, $V^H := \{v \in V \mid h \cdot v = \varepsilon(h)v \ \forall h \in H\}$, the $H$- invariants of $V$.*

*If also $V$ and $W$ are finite-dimensional, then*
*(4)* $\chi_{V\otimes W} = \chi_V * \chi_W$ *and* $\chi_{V^*} = \chi_V \circ S_H = S_{H^*}\chi_V$.

Some general references on Hopf algebras are [M] and [Sch1].

## 2. Frobenius algebras and semisimplicity

In this section we consider the relationship between semisimplicity and the square of the antipode $S$, as well as some useful trace formulas.

To say that a Hopf algebra $H$ is *semisimple* simply means that it is semisimple as an algebra. The next results, due to Larson and Radford, were in fact conjectured by Kaplansky in 1975 [Kap].

**Theorem 2.1.** [LR1] *Assume that $k$ has characteristic 0. Then $S^2 = id$ if and only if both $H$ and $H^*$ are semisimple.*

**Theorem 2.2.** [LR2] *Assume that $k$ has characteristic 0. Then $H$ is semisimple if and only if $H^*$ is semisimple.*

Recently Etingof and Gelaki [EG2] have shown that the "if" direction of Theorem 2.1 also holds in characteristic $p > 0$; Theorem 2.2 is trivially false in that case (consider $H = kG$ such that $p$ divides $|G|$).

Here we give fairly complete proofs of the two results of Larson and Radford. The original proofs of Theorems 2.1 and 2.2 were quite complicated. The first major simplification of the proof of Theorem 2.1 was given by [Sch1]; he used properties of Frobenius algebras in an essential way. We follow the general outline of Schneider's proof, although with a few further small simplifications, including an idea in [Kad]; again, whenever possible, Hopf computations are replaced by working in more general Frobenius algebras. We also give a new shorter proof of Theorem 2.2.

Recall that a $k$-algebra $A$ is a *Frobenius algebra* if there exists an associative, non-degenerate bilinear form $(\ ,\ ) : A \otimes A \to k$. There are a number of equivalent formulations of an algebra being Frobenius. To state these, note that $A^*$ is always a left $A$-module via the transpose of right multiplication in $A$. That is, for $a \in A$ and $f \in A^*$, $a \rightharpoonup f \in A^*$ via $(a \rightharpoonup f)(b) := f(ba)$, for all $b \in A$. Similarly $A^*$ is a right $A$-module via $(f \leftharpoonup a)(b) := f(ab)$. We sometimes write $a \rightharpoonup f = af$ and $f \leftharpoonup a = fa$, when there is no ambiguity.

**Lemma 2.3.** [CR, §9A,B] *Let $A$ be a finite-dimensional algebra. Then the following are equivalent:*

*(1) $A$ is a Frobenius algebra;*

*(2) there exists $f \in A^*$ such that the map $\Phi : {}_AA \to {}_AA^*$, via $a \mapsto (a \rightharpoonup f)$, is a left $A$-module isomorphism;*

*(3) there exists $f \in A^*$ such that the map $\Psi : A_A \to {A^*}_A$, via $a \mapsto (f \leftharpoonup a)$, is a right $A$-module isomorphism;*

*(4) there exists $f \in A^*$ and $r_i, l_i \in A$, $i = 1, ...n$, where $n = dim(A)$, such that for any $a \in A$, $a = \sum_i r_i f(l_i a) = \sum_i f(a r_i) l_i$.*

The set $(f, r_i, l_i)$ is called a *Frobenius system* for $A$. Given such a system, a bilinear form on $A$ is given by $(a, b) := f(ab)$, and similarly $f$ can be reconstructed from the bilinear form. Moreover the elements $\{r_i, l_i\}$ form a dual basis for $( \, , \, )$ in the usual sense; that is, $(l_i, r_j) = \delta_{i,j}$. Of course these dual bases are not unique. However the element $e := \sum_i r_i \otimes l_i$ is uniquely determined by the bilinear form (more generally, this is true for any pair of dual bases for a non-degenerate bilinear form on a vector space).

We first review some basic properties of finite-dimensional Hopf algebras.

A *left integral* in $H$ is an element $\Lambda \in H$ such that $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$; similarly $\Gamma \in H$ is a *right integral* if $\Gamma h = \varepsilon(h)\Gamma$, for all $h \in H$. We let $\int_H^l$ denote the space of left integrals, and $\int_H^r$ the space of right integrals. We will use the notational convention that in $H^*$, a left integral is denoted by $\lambda$ and a right integral by $\gamma$.

If $\int_H^l = \int_H^r$, then $H$ is called *unimodular*; in that case we simply write $\int_H$ for the space of integrals. For example, both $kG$ and $(kG)^*$ are unimodular, for $\Lambda = \Gamma = \sum_{g \in G} g$ in $kG$ and $\lambda = \gamma = p_1$ in $(kG)^*$.

The following result of Larson and Sweedler guarantees that Hopf algebras are always Frobenius.

**Theorem 2.4.** [LS] [M, 2.1.3, 2.2.1] *Let $H$ be any (finite-dimensional) Hopf algebra. Then:*

*(1) $\int_H^l$ and $\int_H^r$ are each one-dimensional;*

*(2) the antipode $S$ of $H$ is bijective;*

*(3) $H^*$ is a cyclic left and right $H$-module, via $\rightharpoonup$ and $\leftharpoonup$;*

*(4) $H$ is a Frobenius algebra;*

*(5) ("Maschke's theorem") $H$ is semisimple $\iff \varepsilon(\Lambda) \neq 0$ for $0 \neq \Lambda \in \int_H^l \iff \varepsilon(\Gamma) \neq 0$ for $0 \neq \Gamma \in \int_H^r$. Moreover when $H$ is semisimple, $H$ is unimodular.*

In the proof of the theorem, the cyclic generators of $H^*$ as a left and right $H$-module are precisely the left (or right) integrals in $H^*$. Moreover, an associative, non-degenerate, bilinear form on $H$ is given by $0 \neq \lambda \in \int_{H^*}^l$ via $(h, l) := \lambda(hl)$ for all $h, l \in H$; another such bilinear form is given by $0 \neq \gamma \in \int_{H^*}^r$ via $(h, l)' := \gamma(hl)$.

Even when $H$ is not unimodular, it will still be true that $\Lambda h \in k\Lambda$, since $\int_H^l$ is one-dimensional by Theorem 2.4. Thus there exists $\alpha \in G(H^*)$ such that $\Lambda h = \alpha(h)\Lambda$ for all $h \in H$; $\alpha$ is called the *left modular function* for $H$. Similarly $\beta = \alpha^{-1}$ is the *right modular function* for $H$. For the Hopf algebra $H^*$, its left and right modular functions can be identified with elements $a, b \in H \cong H^{**}$.

The next lemma is well-known. The second part is the analog of a known result for Haar integrals.

**Lemma 2.5.** *Choose $\lambda \in \int_{H^*}^l$.*

*(1) Let $t \in H$ be such that $\lambda \leftharpoonup t = \varepsilon$. Then $t \in \int_H^r$ and $\lambda(t) = 1$.*

*(2) Let $a$ be the left modular element for $H^*$ in $H$. Then $a\lambda = a \rightharpoonup \lambda \in \int^r_{H^*}$; similarly $\lambda a \in \int^r_{H^*}$.*

*Moreover for any $\Gamma \in \int^r_H$, we have $(a \rightharpoonup \lambda)(\Gamma) = \lambda(\Gamma)$.*

*Proof.* (1) For any $h \in H$,

$$\lambda \leftharpoonup th = (\lambda \leftharpoonup t) \leftharpoonup h = \varepsilon \leftharpoonup h = \varepsilon(h)\varepsilon = \lambda \leftharpoonup (\varepsilon(h)t)$$

and thus $t \in \int^r_H$. Now using that $1 \in H$ is the counit of $H^*$,

$$\lambda(t) = \lambda(t1) = (\lambda \leftharpoonup t)(1) = \varepsilon(1) = 1.$$

(2) Since $\lambda \in \int^l_{H^*}$, for any $f \in H^*$, $f\lambda = f(1)\lambda$ and $\lambda f = f(a)\lambda$. Thus for any $h \in H$, using that $a$ is group-like,

$((a \rightharpoonup \lambda)f)(h) = (a \rightharpoonup (\lambda(a^{-1} \rightharpoonup f))(h) = (\lambda(a^{-1} \rightharpoonup f))(ha)$
$= (a^{-1} \rightharpoonup f)(a)\lambda(ha) = f(aa^{-1})\lambda(ha) = f(1)\lambda(ha) = f(1)(a \rightharpoonup \lambda)(h).$

Thus $(a\lambda)f = f(1)(a\lambda)$ and so $a\lambda$ is a right integral in $H^*$. The argument for $\lambda a$ is similar.

Now if $\Gamma \in \int^r_H$, then $a\lambda(\Gamma) = \lambda(\Gamma a) = \lambda(\Gamma\varepsilon(a)) = \lambda(\Gamma)$, using that $\varepsilon(a) = 1$ since $a \in G(H)$. □

Returning to arbitrary Frobenius algebras, note that any Frobenius algebra $A$ has a natural automorphism $\eta$ determined by the bilinear form, via $(a, b) = (b, \eta(a))$, for all $a, b \in A$; equivalently $f(ab) = f(b\eta(a))$. $\eta$ is called the *Nakayama automorphism* of $A$ relative to ( , ).

A useful technique is to compare different pairs of dual bases, and different forms of the Nakayama automorphism. The next lemma goes back to work of the 1950's and 1960's; for a modern reference, see [Kad, Theorem 1.6].

**Lemma 2.6.** *If $A$ is a Frobenius algebra with Frobenius system $(f, r_i, l_i)$, then any other Frobenius system for $A$ is of the form $(d \rightharpoonup f, r_i d^{-1}, l_i)$ for some unit $d \in A$. The Nakayama automorphism for the new Frobenius system for $A$ is given by $\eta'(a) = d\eta(a)d^{-1}$.*

This lemma was used by Kadison to further shorten Schneider's proof of Theorem 2.10 below. We also use it, in a slightly different way.

The next result was proved under an additional hypothesis in [OS]. The general case is in [Sch1].

**Proposition 2.7.** *Let $H$ be any (finite-dimensional) Hopf algebra. Choose $\lambda \in \int^l_{H^*}$ and $\Gamma \in \int^r_H$ such that $\lambda(\Gamma) = 1$. Then $(\lambda, S\Gamma_1, \Gamma_2)$ is a Frobenius system for $H$. Moreover the Nakayama automorphism $\eta$ for $\lambda$ is given by $\eta(h) = \beta \rightharpoonup S^2(h)$, where $\beta \in H^*$ is the right modular function for $H$.*

Note that such a choice of $\lambda$ and $\Gamma$ can be made using the fact, from Theorem 2.4, that $H^* = \lambda \leftharpoonup H$: choose $\Gamma \in H$ such that $\lambda \leftharpoonup \Gamma = \varepsilon$. Then $\Gamma \in \int^r_H$ and $\lambda(\Gamma) = 1$ by Lemma 2.5.

Using the facts above about Frobenius algebras, we see:

**Corollary 2.8.** [Sch1] *Choose $\gamma \in \int^r_{H^*}$ and $\Gamma \in \int^r_H$ such that $\gamma(\Gamma) = 1$.*

*(1) A Frobenius system for H is* $(\gamma, S^{-1}\Gamma_2, \Gamma_1)$*. Moreover the Nakayama automorphism* $\eta'$ *for* $\gamma$ *is given by* $\eta'(h) = S^{-2}(h) \leftharpoonup \beta$.

*(2) Another Frobenius system for H is* $(\gamma, (S\Gamma_1)a^{-1}, \Gamma_2)$*, where* $a \in H$ *is the left modular element for* $H^*$*. The Nakayama automorphism is also given by* $\eta'(h) = a(\beta \rightharpoonup S^2(h))a^{-1}$.

*Proof.* Part (1) is simply Proposition 2.7 restated for $H^{cop}$.

To see (2), choose $\lambda$ as in the Proposition such that $\lambda(\Gamma) = 1$. Now by Lemma 2.5, $a\lambda \in \int_{H^*}^{r}$, and so $a\lambda = c\gamma$, for some $c \in k$. But $a\lambda(\Gamma) = \lambda(\Gamma) = 1 = \gamma(\Gamma)$, and thus $a\lambda = \gamma$. The corollary now follows by Lemma 2.6 with $d = a$. □

Corollary 2.8 enables one to compute the Nakayama automorphism explicitly. We consider a non-trivial example.

**Example 2.9.** Let $H$ be the Taft Hopf algebra of dimension $n^2$. That is,

$$H = k\langle g, x \mid g^n = 1,\ x^n = 0,\ xg = \omega gx,\ g \in G(H),\ \Delta x = x \otimes 1 + g \otimes x\rangle,$$

where $\omega$ is a primitive $n^{th}$ root of 1. Then $H$ is not unimodular, since $\Lambda = (1 + g + \cdots + g^{n-1})x^{n-1}$ and $\Gamma = x^{n-1}(1 + g + \cdots g^{n-1})$. Since $H$ is self-dual, the left and right integrals in $H^*$ are given similarly. The modular functions on $H$ are given by $\alpha(x) = \beta(x) = 0$ and $\alpha(g) = \omega^{-1}$, $\beta(g) = \omega$. Thus by Proposition 2.7, the Nakayama automorphism for $\lambda$ is given by $\eta(g) = \omega g$ and $\eta(x) = \omega x$; by Corollary 2.8, the Nakayama automorphism for $\gamma$ is $\eta'(g) = \omega g$ and $\eta'(x) = x$.

We remark that this example is important for several reasons. First, if $n = p$, a prime, then the only known non-semisimple Hopf algebras over $\mathbb{C}$ of dimension $p^2$ are of this form. Second, in quantum groups, one can construct Lusztig's "restricted" quantum enveloping algebra $u_q(sl_2)'$, when $q$ is a $2n^{th}$ root of 1, by amalgamating two copies of $H$ in a suitable way.

Comparing the two forms of the Nakayama automorphism for $\gamma$ in Lemma 2.7 enabled Schneider to give a shorter proof of the following crucial result of Radford.

**Theorem 2.10.** [R] *Let H be any finite-dimensional Hopf algebra, with right modular elements* $\beta \in H^*$ *and* $b \in H$*. Then for all* $h \in H$,

$$S^4(h) = b(\beta^{-1} \rightharpoonup h \leftharpoonup \beta)b^{-1}.$$

*Proof.* Since $b$ and $\beta$ are group-like, $S^2(b) = b$ and $S^2$ commutes with $\beta \rightharpoonup$. By Corollary 2.8, using $b = a^{-1}$,

$\eta'(h) = S^{-2}(h) \leftharpoonup \beta = b^{-1}(\beta \rightharpoonup S^2(h))b$.

Applying $S^2$ and conjugating by $b$, we see that $b(h \leftharpoonup \beta)b^{-1} = \beta \rightharpoonup S^4(h)$.

Finally apply $\beta^{-1} \rightharpoonup$ to see that $S^4(h) = b(\beta^{-1} \rightharpoonup h \leftharpoonup \beta)b^{-1}$. □

Now when $H$ and $H^*$ are semisimple, $b = 1$ and $\beta = \varepsilon$ by Theorem 2.4, part (3). Thus $S^4 = id$. To see that in fact $S^2 = id$ in this situation reduces to showing that $-1$ cannot be an eigenvalue for $S^2$ in $End(A)$. To prove this we will need some elementary facts about traces.

Recall that for any finite-dimensional vector space $V$, $V^* \otimes V \cong End_k(V)$ via $(\phi \otimes v)(w) := \phi(w)v$. Under this isomorphism, if $Tr_V : End(V) \to k$ is the usual trace map, then $Tr_V(\phi \otimes v) = \phi(v)$.

**Lemma 2.11.** *Let $A$ be a Frobenius algebra with Frobenius system $(f, r_i, l_i)$. Let $e \in A$ with $e^2 = ce$, for $c \in k$, and let $F \in End_k(eA)$. Then*

$$cTr_{eA}(F) = \sum_i f(F(el_i)r_i).$$

*Proof.* For any $x \in A$, $ex = \sum_i f(exr_i)l_i$. Thus $e^2x = \sum_i f(exr_i)el_i$, and so $cF(ex) = F(cex) = F(e^2x) = \sum_i f(exr_i)F(el_i)$.

Now using the isomorphism $V^* \otimes V \cong End(V)$ above with $V = eA$, we see that $cF$ corresponds to $\sum_i f(-r_i) \otimes F(el_i)$. Thus $cTr_{eA}(F) = \sum_i f(F(el_i)r_i))$. □

Now for any $H$, the left regular representation of $H$ on itself is given by $H \to End(H)$, via $h \mapsto L_h$, where $L_h$ denotes left multiplication by $h$. Let $\chi_H$ denote the character of this representation, that is $\chi_H(h) = Tr_H(L_h)$. We need some elementary properties of $\chi_H$.

**Lemma 2.12.** *Let $\chi_H = Tr_H$ be as above. Then*

*(1) $\chi_H(\Theta) = \varepsilon(\Theta)$, for all $\Theta \in \int_H^L \cup \int_H^r$;*

*(2) $S^2_{H^*}(\chi_H) = \chi_H)$;*

*(3) $\chi_H^2 = dim(H)\chi_H$.*

*Proof.* (1) First choose $\Lambda \in \int_H^l$ and consider $L_\Lambda$, left multiplication by $\Lambda$ on $V = H$. Now $L_\Lambda(h) = \alpha(h)\Lambda$ for all $h \in H$, and thus under the correspondence $V \otimes V^* \cong End(V)$ above, we see that $L_\Lambda$ corresponds to $\alpha \otimes \Lambda$. Thus $Tr_H(L_\Lambda) = \alpha(\Lambda)$. But clearly $\alpha(\Lambda) = \varepsilon(\Lambda)$.

Similarly if $\Gamma \in \int_H^r$, then $L_\Gamma$ corresponds to $\varepsilon \otimes \Gamma$ and so $Tr_H(L_\Gamma) = \varepsilon(\Gamma)$, proving (1).

(2) follows from the fact that $S_H^2$ is an algebra automorphism of $H$. For, $S^2_{H^*}(\chi_H)(h) = \chi_H(S^2_H(h)) = \chi_H(h)$ for any $h \in H$, since $\chi_H$ is the trace of the left regular representation.

To see (3), consider $H$ as a left $H$-module under left multiplication, and let $H_0$ denote the vector space $H$ but with the trivial left $H$-action. Then $H \otimes H_0 \cong H \otimes H$ as left $H$-modules, via $\phi : h \otimes v \mapsto \sum h_1 \otimes h_2 v$ (note $\phi$ is bijective since $\phi^{-1}$ is given by $h \otimes v \mapsto h_1 \otimes Sh_2 v$). □

**Proposition 2.13.** [L] *Let $H$ be a finite-dimensional Hopf algebra, and as before choose $\lambda \in \int_{H^*}^l$ and $\Gamma \in \int_H^r$ with $\lambda(\Gamma) = 1$. Then:*

*(1) $Tr_H(S^2) = \varepsilon(\Gamma)\lambda(1)$;*

*(2) if $S^2 = id$, then $dim(H) = \varepsilon(\Gamma)\lambda(1)$ and $\chi_H = \varepsilon(\Gamma)\lambda$. Consequently if also $dim(H) \neq 0$ in $k$, then $\lambda = \frac{\lambda(1)}{dim(H)}\chi_H$.*

*Proof.* Apply Lemma 2.11 to $H$ with $e = 1$ along with the fact that $(\lambda, S\Gamma_1, \Gamma_2)$ is a Frobenius system (Proposition 2.7) to see that for $F \in End(H)$, $Tr_H(F) = \sum \lambda(F(\Gamma_2 S\Gamma_1)$.

Now to prove (1), use $F = S^2$ in this expression:

$$Tr(S^2) = \sum \lambda(S^2(\Gamma_2)S(\Gamma_1)) = \sum \lambda(S(\Gamma_1 S(\Gamma_2)) = \lambda(S(\varepsilon(\Gamma)1) = \varepsilon(\Gamma)\lambda(1).$$

Here we have used the property of $S$ that $id * S = \varepsilon$, applied to $\Gamma$.

For (2), assume $S^2 = \text{id}$. Then $dim(H) = \varepsilon(\Gamma)\lambda(1)$ follows directly from (1). For the next equality, first note that if $S^2 = id$, then $S = S^{-1}$ and thus for all $h \in H$, $\sum h_2 S(h_1) = \varepsilon(h)$. Thus, again using $e = 1$ but now with $F = L_h$, we see

$$\chi_H(h) = Tr_H(L_h) = \sum \lambda(h\Gamma_2 S(\Gamma_1)) = \varepsilon(\Gamma)\lambda(h).$$

Thus $\chi_H = \varepsilon(\Gamma)\lambda$. The expression for $\lambda$ now follows by multiplying by $\lambda(1)$ and dividing by $dim(H)$. □

We require one more fact.

**Proposition 2.14.** [LR1] *Choose $\gamma \in \int_{H^*}^r$ and $\Lambda \in \int_H^l$ such that $\gamma(\Lambda) = 1$. Then*

$$Tr_{H^*}(S^2_{H^*}) = \varepsilon(\Lambda)\gamma(1) = dim(H)Tr_{H^*}(S^2|_{\chi_H H^*}).$$

*Proof.* The first equality is simply Corollary 2.13(1) applied to $H^*$. For the second, we apply Lemma 2.11 to $H^*$ with $e = \chi_H$, $c = \dim(H)$, and $F = S^2|_{\chi_H H^*}$, using the fact that $(\Lambda, S\gamma_1, \gamma_2)$ is a Frobenius system for $H^*$. By Lemma 2.12(3), $e^2 = ce$. Thus,

$$dim(H)Tr_{H^*}(S^2|_{\chi_H H^*}) = \sum \Lambda(S^2(\chi_H\gamma_2)S\gamma_1) = \sum \Lambda(\chi_H(S^2\gamma_2)S\gamma_1)$$
$$= \sum \Lambda(\chi_H S(\gamma_1 S\gamma_2)) = \Lambda(\chi_H\gamma(1)) = \gamma(1)\chi_H(\Lambda).$$

Now using Lemma 2.12(1), $\chi_H(\Lambda) = \varepsilon(\Lambda)$. This finishes the proof. □

We are now able to prove the first Larson-Radford theorem.

*Proof.* (*of Theorem 2.1*) First, if $S^2 = id$, then by Proposition 2.13(1), both $\varepsilon(\Gamma) \neq 0$ and $\lambda(1) \neq 0$. Thus $H$ and $H^*$ are semisimple by Theorem 2.4(5).

We may thus assume that both $H$ and $H^*$ are semisimple. Then they are also unimodular, and so by Theorem 2.10, $S^4 = id$, as noted after that theorem. Thus $S^2_{H^*}$ has only 1 and -1 as possible eigenvalues on $H^*$. Let $\{\mu_j,\ j = 1, \ldots, n = dim(H)\}$ be the eigenvalues of $S^2_{H^*}$ on $H^*$ and let $\{\eta_i,\ i = 1, \ldots, m\}$ be the eigenvalues of $S^2|_{\chi_H H^*}$. By Proposition 2.14, $\sum_{j=1}^n \mu_j = n(\sum_{i=1}^m \eta_i)$.

Now $\sum \mu_j = Tr_{H^*}(S^2) \neq 0$ by Proposition 2.13(1), and thus it is an integer divisible by $n$. Thus $|\sum_{j=1}^n \mu_j| = n$. But at least one $\mu_j = 1$, since $S^2(\varepsilon) = \varepsilon$. Thus all $\mu_j = 1$ and $S^2 = \text{id}$. □

We can now also prove the second Larson-Radford theorem. We use the following elementary fact about matrices over $\mathbb{C}$.

**Lemma 2.15.** *Let $k$ be algebraically closed of characteristic 0 and let $A = M_n(k)$, the algebra of $n \times n$ matrices over $k$. Let $T \in Aut_k(A)$ be such that $T^m = id$, for some $m > 0$. Then $Tr_A(T)$ is a non-negative real number.*

*Proof.* (*of Theorem 2.2*) By extending the base field, we may assume $k$ is algebraically closed. Now assume $H$ is semisimple. Let $T = S^2$; $T$ is an automorphism of $H$ since $S$ is an anti-automorphism. By Theorem 2.10, $T$ has finite order, since $\alpha = \varepsilon$ and $b$ has finite order (group-like elements in any finite-dimensional Hopf algebra have finite order, since distinct group-like elements are linearly independent).

Now $H = \oplus M_{n_i}(k) = \oplus A_i$, so $T$ permutes the $\{A_i\}$. We claim that $Tr_H(T) \neq 0$. Now if $T(A_i) = A_j$, for $j \neq i$, let $r > 0$ be minimal with $T^r(A_i) = A_i$. Letting $B = \oplus_{s=0}^{r-1} T^s(A_i)$, we see that $Tr_H(T|_B) = 0$. If $T$ stabilizes $A_i$, then $T \in Aut_k(A_i)$ and so by Lemma 2.15, $Tr_H(T|_{A_i}) \geq 0$. Thus $Tr_H(T) \geq 0$.

However, $H$ has a one-dimensional summand, say $A_0$, generated by $\int_H$. Since $T$ is an automorphism of $A_0$, $T|_{A_0} = id$ and so $Tr_H(T|_{A_0}) = 1$. Thus $Tr_H(T) > 0$.

We are now done by Proposition 2.13(1): $Tr_H(T) = \varepsilon(\Lambda)\lambda(1)$ for $\Lambda \in \int_H^l$ and $\gamma \in \int_{H^*}^r$. Thus $\gamma(1) \neq 0$ and so $H^*$ is semisimple by Maschke's theorem (Theorem 2.4(5)). □

## 3. Character Theory and the Class Equation

We assume from now on that $H$ is semisimple and that $k$ has characteristic 0. Thus $S^2 = \text{id}$ by Theorems 2.1 and 2.2, and so we may assume that $V^{**} = V$ for any finite-dimensional $H$-module $V$. Let $V_0, V_1, \dots, V_m$ be a complete set of irreducible left $H$-modules, where $V_0 = kv_0$ denotes the trivial module as in §1. Note that $V_0 \cong k\Lambda$ for any $0 \neq \Lambda \in \int_H$.

Let $n_i = dim_k(V_i)$ and let $\phi_i : H \to End(V_i) \cong M_{n_i}(k)$ be the $i^{th}$ irreducible representation of $H$, with character $\chi_i$. $n_i$ is called the *degree* of $\chi_i$ (and of $\phi_i$). Note that we may write $\chi_H = \sum_{i=0}^m n_i \chi_i$.

It is easy to see that $V_i^*$ is also an irreducible left $H$-module; we sometimes write $\chi_{i^*}$ for $\chi_{v_i^*} = S\chi_i$.

Since $H$ is semisimple, any $H$-module is a sum of the $V_i$, with multiplicities. In particular

$$V_i \otimes V_j \cong \sum_l m_{ij}(l) V_l \tag{3.0}$$

where the multiplicities $m_{ij}(l)$ are non-negative integers.

**Definition 3.1.** The *character algebra* $\mathcal{R}(H)$ of $H$ is the $k$-span in $H^*$ of all the characters on $H$.

$\mathcal{R}(H)$ is spanned by $\{\chi_0, \dots, \chi_m\}$, and in fact they form a basis.

We note that for $f \in H^*$, $f \in \mathcal{R}(H)$ if and only if $f(hl) = f(lh)$ for all $h, l \in H$. This is clear if $f \in \mathcal{R}(H)$; the converse follows since $H$ is a sum of matrix rings and the only linear functional on matrices with this property is the trace, up to a scalar multiple.

Since $\Delta_{H^*}$ is the transpose of multiplication $m_H$, this condition may be rewritten to say that $f \in \mathcal{R}(H)$ if and only if $\Delta_{H^*} f = \sum f_1 \otimes f_2 = \sum f_2 \otimes f_1$; that is, $f$ is a "cocommutative element" of $H^*$.

The analog of orthogonality of characters is true (an old result of Larson). First, we define a bilinear form on $H^*$, generalizing the one for functions on groups, as in [Se].

**Definition 3.2.** Choose $\Lambda \in \int_H$ with $\varepsilon(\Lambda) = 1$. For $\phi, \psi \in H^*$, let

$$(\phi|\psi) := (\phi * S\psi)(\Lambda) = \sum_{\Lambda} \phi(\Lambda_1) S\psi(\Lambda_2) = \tilde{\Lambda}(\phi * S\psi),$$

where $\tilde{\Lambda}$ is the element corresponding to $\Lambda$ in $H^{**} \cong H$ and $*$ denotes the convolution product in $H^*$.

**Theorem 3.3.** [L] *(Orthogonality of Characters) Let $H$ be semisimple. Then $(\chi_i|\chi_j) = \delta_{ij}$.*

*Proof.* Since $V_0 \cong k\Lambda$, $\chi_0(\Lambda) = \varepsilon(\Lambda)$ and $\chi_i(\Lambda) = 0$ for $i > 0$. The relation (3.0), replacing $V_j$ by $V_j^*$, gives the corresponding relation on characters $\chi_i * \chi_{j^*} = \sum_l m_{ij^*}(l)\chi_l$. Applying this expression to $\Lambda$ gives

$$\sum \chi_i(\Lambda_1) S\chi_j(\Lambda_2) = \sum_l m_{ij^*}(l)\chi_l(\Lambda) = m_{ij^*}(0)\varepsilon(\Lambda).$$

Now $m_{ij^*}(0)$ is the multiplicity of the trivial representation in $V_i \otimes V_j^*$. By Lemma 1.1(2)(3), this is the same as the multiplicity of the trivial representation in $Hom_H(V_j, V_i)$. Thus $m_{ij^*}(0) = dim(Hom_H(V_j, V_i))$. By Schur's Lemma, $Hom_H(V_j, V_i) = 0$ if $i \neq j$ and $Hom_H(V_i, V_i) \cong k$. Thus $m_{ij^*}(0) = \delta_{ij}$, proving the theorem. □

As an application of the orthogonality theorem, one may prove a number of relations on the multiplicities in 3.0. We give one example, from [NR], in (3) of the next lemma. The present proof follows the classical one for groups, and seems shorter.

**Corollary 3.4.** *(1) $(\ |\ )$ is a symmetric $S$-invariant form on $\mathcal{R}(H)$ .*

*(2) For any $i, j, l$, $m_{ij}(l) = (\chi_l|\chi_i\chi_j)$.*

*(3) For any $i, j, l$, the following multiplicity relation holds:*
*$m_{jl}(i) = m_{li^*}(j^*) = m_{il^*}(j)$.*

*Proof.* (1) It suffices to prove for the $\chi_i$, as they form a basis of $\mathcal{R}(H)$. It is clear from Theorem 3.3 that the form is symmetric on the $\chi_i$. The $S$-invariance also follows from Theorem 3.3, since $S\chi_V = \chi_{V^*}$ and $V_i^* \cong V_j^* \iff V_i \cong V_j$.

(2) This follows immediately from the Theorem and (3.0).

(3) We use (1) and (2). Note that it follows from the definition that $(\phi|\psi) = (\phi S(\psi)|\varepsilon)$. Now
$m_{jl}(i) = (\chi_i|\chi_j\chi_l) = (\chi_i S(\chi_j\chi_l)|\varepsilon) = (\chi_i S\chi_l S\chi_j|\varepsilon) = (S(\chi_l S\chi_i)S\chi_j|\varepsilon)$
$=(\chi_l S\chi_i|S\chi_j) = m_{li^*}(j^*)$.
The second relation is similar. □

The proof of the next lemma is close to an old argument for group algebras.

**Lemma 3.5.** [Kac, Z] *$\mathcal{R}(H)$ is a semisimple algebra with involution $*$.*

*Proof.* The antipode $S$ is an involution on $\mathcal{R}(H)$, since $S^2 = \mathrm{id}$ and $S\chi_i = \chi_{i^*}$. It suffices to prove semisimplicity for $\mathcal{R}_{\mathbb{Q}}(H)$, the $\mathbb{Q}$-linear span of the $\chi_i$, since $\mathcal{R}(H)$ is obtained by extending the base field.

We first claim that for any $0 \neq \lambda \in \mathcal{R}_{\mathbb{Q}}(H)$, $\lambda\lambda^* \neq 0$. For, write $\lambda = \sum_i a_i\chi_i$, where $a_i \in \mathbb{Q}$. Then

$$\lambda\lambda^* = \sum_{ij} a_i a_j \chi_i \chi_{j^*} = \sum a_i a_j (\delta_{ij}\varepsilon + \sum_{l\geq 1} m_{ij^*}(l)\chi_l) = (\sum_i a_i^2)\varepsilon + \sum_{l\geq 1} b_l\chi_l$$

for some $b_l \in \mathbb{Q}$. Thus $\lambda\lambda^* \neq 0$.

Now if $\mathcal{R}_{\mathbb{Q}}(H)$ is not semisimple, its Jacobson radical $J \neq 0$ is nilpotent. Choose $0 \neq \lambda \in J$. Then by the above, $0 \neq \gamma = \lambda\lambda^* \in J$. But $\gamma^* = \gamma$ and so again by the above computation, $\gamma^2 \neq 0$. Repeating this, we see that $\gamma^t \neq 0$ for all $t$, a contradiction. Thus $J = 0$ and we are done. □

The following theorem, known as the "Class Equation" for Hopf algebras, was shown by G. I. Kac for Hopf $C^*$-algebras and in general by Y. Zhu.

**Theorem 3.6.** [Kac, Z] *Let $H$ be a finite-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic 0. Let $\{e_0, e_1, \ldots, e_m\}$ be a complete set of primitive orthogonal idempotents in $\mathcal{R}(H)$, where $e_0$ is an integral in $H^*$. Then for each $i$, $dim(e_iH^*)$ divides $dim(H)$, and*

$$dim(H) = 1 + \sum_{i=1}^{m} dim(e_iH^*)$$

We give Lorenz' recent representation-theoretic proof of the theorem [Lo]. The first lemma is well-known; see for example [CR, p. 204], although the present proof seems shorter.

**Lemma 3.7.** *Let $k$ be a field of characteristic 0 and let $A$ be a split semisimple $k$-algebra. That is, $A = \oplus_{j=1}^{d} A_i$, $A_i \cong M_{n_i}(k)$. Assume $\langle\ |\ \rangle$ is a non-degenerate symmetric associative bilinear form on $A$ and let $\{a_j, b_j\}$, $j = 1, \ldots, n$ be dual bases for $A$ with respect to $\langle\ |\ \rangle$. For each $i$, set*

$$f_i = \sum_{j=1}^{n} tr_i(a_j)b_j,$$

*where $tr_i$ is the usual matrix trace in $A_i$. If $\tilde{e}_i$ is the primitive central idempotent corresponding to $A_i$, then $f_i = \alpha_i\tilde{e}_i$, for some $0 \neq \alpha_i \in k$.*

*Proof.* Since $\langle\ |\ \rangle$ is symmetric and associative, $f(a) := \langle 1|a\rangle$ satisfies $f(ab) = f(ba)$, for all $a, b \in A$. It follows that for some scalars $\gamma_i \in k$, $< a|b >= \sum_{i=1}^{d} \gamma_i tr_i(ab)$, where the $\gamma_i \neq 0$ since the form is non-degenerate. One set of dual bases is given by $\{\gamma_i^{-1}e^i_{lm}, e^i_{ml}\}$, where for each $i$, $\{e^i_{lm}\}$ is a set of matrix units for the $i$th summand $M_{n_i}(k)$. But has been noted earlier, for any two pair of dual bases $\{a_j, b_j\}$ and $\{c_r, d_r\}$ for $A$ with respect to the form $\langle \mid \rangle$, it is always true that $\sum_j a_j \otimes b_j = \sum_r c_r \otimes d_r$.

It now follows that

$$f_i = \sum_{j,l,m} \gamma_j^{-1} tr_i(e^j_{lm}) e^j_{ml} = \sum_m \gamma_i^{-1} tr_i(e^i_{mm}) e^i_{mm} = \gamma_i^{-1} \tilde{e}_i.$$

Note that $\alpha_i = \gamma_i^{-1}$. □

To apply this lemma, we need a new form on $\mathcal{R}(H)$, since the form in 3.2 is not associative.

**Definition 3.8.** *Choose* $\Lambda \in \int_H$ *with* $\varepsilon(\Lambda) = 1$. *For* $\phi, \psi \in \mathcal{R}(H)$, *let*

$$\langle \phi | \psi \rangle := (\phi * \psi)(\Lambda) = \tilde{\Lambda}(\phi * \psi),$$

*where as in 3.2,* $\tilde{\Lambda} \in H^{**}$ *corresponds to* $\Lambda$.

The two forms are closely related: $\langle \phi | \psi \rangle = (\phi | S\psi)$. But now, $\langle \mid \rangle$ is clearly associative on $\mathcal{R}(H)$; moreover it is symmetric since $( \mid )$ is symmetric and $S$-invariant (alternatively, $\tilde{\Lambda}(\phi\psi) = \tilde{\Lambda}(\psi\phi)$ since $\tilde{\Lambda} \in \mathcal{R}(H)$ by Proposition 2.13(2)). In addition, it follows from Theorem 3.3 that $\{S\chi_j, \chi_j\}$ are dual bases for $\langle \mid \rangle$.

We can now finish the proof of the Class Equation.

*Proof.* (*of Theorem 3.6*) [Lo] Choose $\lambda \in \int_{H^*}$ with $\lambda(1) = 1$. By Proposition 2.13(2), $\lambda = (1/dim(H))\chi_H \in \mathcal{R}(H)$, so in fact we may choose $e_0 = \lambda$.

Consider $A = \mathcal{R}(H)$ and fix $e = e_i$ a primitive idempotent in $A$. Set $d := \frac{dim(H)}{dim(eH^*)} \in \mathbb{Q}$. It suffices to show $d \in \mathcal{A}$, the ring of algebraic integers in $k$, for then $d \in \mathbb{Z}$.

Let $\tilde{e} = \tilde{e}_i$ be the central primitive idempotent of $A$ to which $e$ belongs. Then $\tilde{e}\mathcal{R}(H) \cong M_{m_i}(k) \cong (e_i\mathcal{R}(H))^{(m_i)}$ and so $dim(\tilde{e}H^*) = m_i dim(eH^*)$. Thus $d = \frac{m_i(dim(H^*))}{dim(\tilde{e}H^*)}$.

Define $f_i := \sum_j tr_i(S\chi_j)\chi_j$. Then by the previous lemma, $f_i = \alpha_i \tilde{e}_i$.

We first claim that $d = \alpha_i$. Now by Proposition 2.13(2) applied to $H^*$, with $\Lambda$ as in 3.8, $Tr_{H^*} = (dim(H))\Lambda$. Thus $dim(\tilde{e}H^*) = Tr_{H^*}(L_{\tilde{e}}) = (dim(H))\tilde{e}(\Lambda) = (dim(H))\alpha_i^{-1} f_i(\Lambda)$. But

$$f_i(\Lambda) = \sum_j tr_i(S\chi_j)\chi_j(\Lambda) = tr_i(\varepsilon)\varepsilon(\Lambda) = m_i.$$

Here we have used again (as in the proof of Theorem 3.3), that $\chi_0 = \varepsilon$ and that $\chi_j(\Lambda) = \delta_{j0}\varepsilon(\Lambda)$. Thus $dim(\tilde{e}H^*) = dim(H)\alpha_i^{-1} m_i$. It follows by the second formula for $d$ that $d = \alpha_i$. Thus $f_i = d\tilde{e}_i$.

Finally we show that $f_i$ is integral over $\mathbb{Z}$. This will finish the proof of the theorem, for then, since $\tilde{e}$ is an idempotent, $f_i^r = d^r \tilde{e}$ for all $r \geq 0$ and so $d$ is integral over $\mathbb{Z}$.

Consider the ring $\mathcal{R}_{\mathbb{Z}}(H) := \sum_i \mathbb{Z}\chi_i \subset \mathcal{R}(H)$. Since the $\{\chi_i\}$ are $\mathbb{Z}$-independent by orthogonality, $\mathcal{R}_{\mathbb{Z}}(H)$ is a finite free $\mathbb{Z}$-module, and so any

$x \in \mathcal{R}_{\mathbb{Z}}(H)$ is integral over $\mathbb{Z}$. If $\pi_i : \mathcal{R}(H) \to M_{m_i}(k)$ is the $i^{th}$ irreducible representation of $\mathcal{R}(H)$, then also $\pi_i(x)$ is integral over $\mathbb{Z}$, as are all of its eigenvalues, and so $tr_i(x)$ is integral over $\mathbb{Z}$.

In particular, letting $x = \chi_j$, we see that $tr_i(\chi_j) \in \mathcal{A}$, the ring of algebraic integers, for all $j$. Thus $f_i \in \sum_j \mathcal{A}\chi_j \subset \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{R}_{\mathbb{Z}}(H)$, which is also integral over $\mathbb{Z}$. □

When $H = kG$, the theorem reduces to the usual class equation for finite groups. For, $f \in \mathcal{R}(H) \iff f(ghg^{-1}) = f(h),\ \forall g \in G \iff f$ is constant on the conjugacy classes of $G$. Thus $e_i = \sum_{g \in C_i} p_g$, where $C_i$ is the $i^{th}$ conjugacy class. Then $dim(e_i H^*) = |C_i|$ and this divides $|G|$.

When $H = (kG)^*$, Theorem 3.6 specializes to Frobenius' classical theorem that the degree of an irreducible $G$-module divides $|G|$. For in this case, $H \cong k^{(|G|)}$ and so there are $|G|$ linearly independent characters. Thus $\mathcal{R}(H) = H^* = kG$, and the $e_i$ are just the primitive idempotents of $kG$ and the $e_i H^*$ are the irreducible $kG$-modules.

The analog of Frobenius' theorem for Hopf algebras was conjectured by Kaplansky in 1975 and remains open. That is,

**Conjecture 3.9.** [Kap] *Let $H$ be a finite-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then for any irreducible module $V$ of $H$, $dim(V) \mid dim(H)$.*

We discuss this conjecture further in the next section.

## 4. Applications of the Class Equation

We assume throughout that $k$ is algebraically closed of characteristic 0. Our first application of the Class Equation is the one for which it was proved - the classification of Hopf algebras of prime dimension.

**Theorem 4.1.** [Kac, Z] *If $\dim H = p$, a prime, then $H \cong k\mathbb{Z}_p$.*

*Proof.* By the Class Equation 3.6, $p = 1 + \sum_{i>0} p^{m_i}$. This forces $m_i = 0$ for all $i$. Consequently there are $p$ linearly independent characters $\chi_i$ of $H$. This implies that $H \cong k^{(p)}$ and that the characters are algebra maps. Dually, this means that $H^*$ has a basis of group-like elements, that is, $H^* \cong kG \cong k\mathbb{Z}_p$. But then $H$ is self-dual and so $H \cong k\mathbb{Z}_p$. □

In fact this result does not require that we assume $H$ is semisimple: for, one may assume that both $H$ and $H^*$ are unimodular, since otherwise the modular elements $\alpha$ and $a$ would be non-trivial group-like elements, necessarily of order $p$, and we would be done. Thus by Theorem 2.10, $S^4 = \mathrm{id} = (S^2)^2$, and so $S^2$ has eigenvalues in $\{1, -1\}$. Writing $H = H_+ \oplus H_-$, the eigenspaces for $H$, we see that $Tr_H S^2 = dim(H_+) - dim(H_-)$. By Proposition 2.13(1), $Tr_H S^2 = 0$ if $H$ is not semisimple. But then $dim(H) = 2dim(H_+)$, a contradiction unless $p = 2$. This case is trivial.

More generally, the Class Equation can be used to show that if $dim(H) = p^n$, for $p$ a prime, then $H$ contains a central group-like element [Kac, Ma2]. It follows from this that if $\dim(H) = p^2$, then $H \cong kG$ [Ma2].

Our next application is the analog of a well-known fact for finite groups: if $K$ is a subgroup of $G$ of index the smallest prime divisor of $|G|$, then $K$ is normal in $G$.

**Theorem 4.2.** [KM] *Let $H$ be semisimple, and let $K$ be a Hopf subalgebra of $H$ such that the index $[H : K] := \frac{dim(H)}{dim(K)}$ is the smallest prime divisor of $dim(H)$. Then $K$ is normal in $H$.*

Before giving a proof, we need some background. First, the notion of index makes sense, since by the Nichols-Zoeller theorem [M, 3.1.3], $H$ is free over $K$ and so $dim(K)|dim(H)$. This was improved by Schneider to say that $H \cong H/HK^+ \otimes K$ as right $K$-modules [M, 3.3.1, 8.4.6], where $K^+ := K \cap Ker(\varepsilon)$. Thus $dim(H/HK^+) = p$ in our case, and so $dim(HK^+) = dim(H) - p$.

A Hopf subalgebra $K$ of $H$ is *normal* if for all $h \in H$ and $l \in K$, $(ad_h)(l) = \sum h_1 l S(h_2) \in K$. When $H$ is finite-dimensional, one can show that $K$ is normal if and only if $HK^+ = K^+H$ [M, 3.4.4].

We give a proof based on work of Natale [Na, Theorem 2.1.1].

*Proof.* Let $\Lambda \in \int_H$ and $e \in \int_K$, with $\varepsilon(\Lambda) = \varepsilon(e) = 1$. By Proposition 2.13(2), $e \in \mathcal{R}(K^*) \subset K$ and so $e$ is a cocommutative element in $K \subset H$. But then also $e \in \mathcal{R}(H^*) \subset H$. Thus $e = \Lambda + e_1 + \cdots + e_s$, where the $\{e_i\}$ are primitive orthogonal idempotents of $\mathcal{R}(H^*)$; note that $\Lambda$ appears in this sum since $\Lambda e = \Lambda$. Thus $He = H\Lambda \oplus He_1 \oplus \cdots \oplus He_s$.

Now $HK^+ = H(1-e)$ and so $H = He \oplus H(1-e) = He \oplus HK^+$. Thus by the remarks before the proof, $dim(He) = p$. But $dim(H\Lambda) = 1$ and so by the Class Equation, $dim(He_i)$ divides $dim(H)$ but is less than $p$. Thus $dim(He_i) = 1$ and so $e_i$ is central, for all $i$. Thus $e$ is also central in $H$. It follows that $HK^+ = K^+H$ and so $K$ is normal in $H$. □

For our third application, we give Schneider's recent more elementary proof of the Etingof-Gelaki theorem [EG] on the dimensions of irreducible $D(H)$-modules, where $D(H)$ is the Drinfel'd double of $H$; the original proof used some non-trivial facts about modular categories.

**Theorem 4.3.** [EG] *Let $H$ be a semisimple Hopf algebra and let $D(H)$ denote its Drinfel'd double. Then for any irreducible module $V$ of $D(H)$, $dim(V) \mid dim(H)$.*

Before giving Schneider's proof, we remind the reader of the definition of $D(H)$. We follow the notation in [M, §10.3]. As a $k$-coalgebra, $D(H) = H^{*cop} \otimes H$, and the elementary tensors in $D(H)$ are usually written as $f \otimes h = f \bowtie h$. Thus

$$\Delta_D(f \bowtie h) = \sum (f_2 \bowtie h_1) \otimes (f_1 \bowtie h_2)$$

and $\varepsilon_D = 1_H \otimes \varepsilon_H$. Multiplication in $D(H)$ is more complicated and involves the coadjoint actions of $H$ and $H^*$ on each other. We give an alternative formula of Radford (see [M, 10.3.11]). That is, for $f, f' \in H^*$ and $h, h' \in H$,

$$(f \bowtie h)(f' \bowtie h') = \sum f(h_1 \rightharpoonup f' \leftharpoonup S^{-1}h_3) \bowtie h_2h'.$$

Clearly $1_D = \varepsilon_H \bowtie 1_H$. For simplicity we write $h = (\varepsilon \bowtie h)$ and $f = (f \bowtie 1)$. Note that both $H \cong \varepsilon \bowtie H$ and $H^{*cop} \cong H^{*cop} \bowtie 1$ as Hopf algebras.

For example, if $H = kG$, and $g, g', h, h' \in G$, then multiplication is

$$(p_g \bowtie h)(p_{g'} \bowtie h') = \sum p_g(h \rightharpoonup p_{g'} \leftharpoonup h^{-1}) \bowtie hh' = \sum p_g p_{hg'h^{-1}} \bowtie hh'.$$

Thus $D(H) \cong (kG)^* \# kG$ as algebras, a skew group ring over $(kG)^*$.

Drinfel'd shows that $D(H)$ is quasi-triangular, with "$R$-matrix" given as follows: let $\{h_i, f_i\}$ be dual bases of $H$ and $H^*$ under the usual evaluation map. Then

$$R = \sum_i (\varepsilon_H \bowtie h_i) \otimes (f_i \bowtie 1_H) \in D(H) \otimes D(H).$$

We write $R = \sum_i h_i \otimes f_i$ for short. By $R^{21}$ we mean the flip map $\tau$ applied to $R$; that is, $R^{21} = \tau(R) = \sum f_i \otimes h_i$. The reader is referred to [Ksl] or [M] for definitions and properties of quasi-triangular Hopf algebras. We note that an important property of any quasi-triangular Hopf algebra $(H, R)$ is that $H$ is "almost cocommutative"; that is, $\tau \circ \Delta(h) = R\Delta(h)R^{-1}$ for all $h \in H$. Consequently $V \otimes W \cong W \otimes V$ for any $V, W \in {}_H\mathcal{M}$ (see [M, 10.1.2]) and thus $\mathcal{R}(H)$ is a commutative ring.

We require several known results:

**Theorem 4.4.** [R2][M, 10.3.12] *If* $0 \neq \Gamma \in \int_H^r$ *and* $0 \neq \lambda \in \int_{H^*}^l$, *then* $D(H)$ *is unimodular and* $\lambda \bowtie \Gamma \in \int_{D(H)}$.

**Lemma 4.5.** [R2, Dr] *Let* $D = D(H)$ *and consider* $F : D^* \to D$ *given by* $p \mapsto (1 \otimes p)(R^{21}R) = \sum_{ij}(f_j \bowtie h_i)p(f_i \bowtie h_j)$. *Then* $F$ *is a vector space isomorphism which takes* $\mathcal{R}(D)$ *to* $Z(D)$, *the center of* $D(H)$.

*Moreover* $F(pq) = F(p)F(q)$ *for all* $p \in D^*, q \in \mathcal{R}(D)$.

In fact [R2, Dr] only note that $F$ is multiplicative on $\mathcal{R}(D)$; Schneider shows that one of the arguments can be chosen arbitrarily.

*Proof.* (*of Theorem 4.3*) (Schneider [Sch2]) Again write $D$ for $D(H)$ for the sake of simplicity. Let $\overline{e}$ be a primitive (central) idempotent of $Z(D)$ and let $e$ be the primitive idempotent of $\mathcal{R}(D)$ such that $F(e) = \overline{e}$. Then by the lemma, $F(D^*e) = F(D^*)\overline{e} = D\overline{e}$. Thus $dim(D^*e) = dim(D\overline{e})$.

Now by the Class Equation 3.6, $dim(D^*e)$ divides $dim(D) = (dim(H))^2$. But $D\overline{e}$ is a full matrix summand of $D$, corresponding to the irreducible module $V$ of $D$. Thus $dim(D\overline{e}) = (dim(V))^2$, and so $dim(V) \mid dim(H)$. □

**Corollary 4.6.** [EG] *Assume* $H$ *is any semisimple quasitriangular Hopf algebra, and let* $V$ *be an irreducible* $H$*-module. Then* $dim(V)$ *divides* $dim(H)$.

*Proof.* This follows from Theorem 4.3 and the known (and fairly easy) fact that a Hopf algebra $H$ is quasitriangular if and only if the map $D(H) \to H$, given by $f \bowtie h \mapsto (f \otimes id)(R)h$, is an algebra morphism. □

.

More generally, Schneider proves a generalization of this result to what are called "factorizable" Hopf algebras [Sch2].

The Corollary shows that Kaplansky's conjecture 3.9 holds if $H$ is quasitriangular. Earlier, two other results about the conjecture were known:
(1) [NR] show that if $dim(V) = 2$, then $2 \mid dim(H)$.
(2) [MW] show that if $H$ is semisolvable, that is, if $H$ has a normal series such that each Hopf quotient in the series is commutative or cocommutative, then Conjecture 3.9 holds.

It would be very useful in classifying Hopf algebras if the conjecture were known. In fact Etingof and Gelaki used 4.6 to prove that any (semisimple) Hopf algebra of dimension $pq$, where $p$ and $q$ are primes, is a group algebra or its dual [EG3]. It had been proved earlier in [GW] that such a result held, provided Conjecture 3.9 was true for these Hopf algebras.

Recently a shorter proof of the $pq$ result has been given by Natale [Na], although still using Theorem 4.3.

## 5. The Grothendieck ring and $K_0(H)$

The usual Grothendieck group $G(H)$ becomes a ring using tensor products of modules (an observation going back to Serre). When $H$ is semisimple, all modules are projective and so $G(H) = K_0(H)$ and has as a basis the isomorphism classes of the irreducible modules. It follows that $K_0(H) \cong \mathcal{R}_{\mathbb{Z}}(H)$ as rings.

We first discuss here work of Nikshych on the relationship of the the ring structure of $K_0$ to the isomorphism class of $H$. To do this we need the notion of (dual) *cocycle twists*. Such twists were introduced by Drinfel'd [Dr2] and by Reshetikhin [Re] in their work on quasi-Hopf algebras.

Consider a Hopf algebra $H$ and an invertible element $\Omega \in H \otimes H$. $H^\Omega$ is called a *cocycle deformation*, or *twist* of $H$ with respect to $\Omega$ if:

(i) $H^\Omega = H$ as an algebra

(ii) $H^\Omega$ has comultiplication $\Delta^\Omega(h) := \Omega(\Delta h)\Omega^{-1}$

One can write down necessary and sufficient conditions on $\Omega$ for $\Delta^\Omega$ to be coassociative; such an element $\Omega$ is called a *pseudo-cocycle*. An important special case of this is when $\Omega$ is an ordinary Hopf 2-cocycle on $H^*$; this means that

$$(\Omega \otimes 1)[(\Delta \otimes id)(\Omega)] = (1 \otimes \Omega)[(id \otimes \Delta)(\Omega)] \tag{5.0}$$

See (5.2) below for the usual (dual) form of this relation.

A nice connection between twisting and $K_0(H)$ has been made by Nikshych.

**Theorem 5.1.** [N] *Let $H_1$ and $H_2$ be two finite-dimensional semisimple Hopf algebras over $\mathbb{C}$. Then $H_1$ is a twist of $H_2$ by a (dual) pseudo-cocycle* *if and only if* $K_0(H_1) \cong K_0(H_2)$ *as ordered rings with involution with a marked element.*

Here the marked element can be taken to be $\chi_H$.

However, having isomorphic $K_0$-rings is not sufficient to guarantee that two Hopf algebras have isomorphic categories of representations. For, consider the non-commutative (semisimple) Hopf algebras of dimension 8. According to [Ma1], there are exactly three such Hopf algebras, namely the two group algebras $kD_4$ and $kQ$ of the dihedral group and the quaternion group, and $H_8$, the Kac-Paljutikin non-commutative, non-cocommutative Hopf algebra of dimension 8, constructed in [KacP] (we consider this example in more detail in the next section). Clearly they are all isomorphic as algebras, since $k^{(4)} \oplus M_2(k)$ is the only possibliity which has a one-dimensional summand. It is not difficult to see that their $K_0$-rings are isomorphic, and thus by Theorem 5.1 the three Hopf algebras are twists of each other by a pseudo-cocycle. However it is shown in [TY] that these three Hopf algebras have different categories of representations, considered as monoidal categories.

Nevertheless, $K_0(H)$ is a useful invariant of $H$. Kashina has classified the Hopf algebras of dimension 16 by first classifying all the possible $K_0$-rings for such algebras [Ksh]. The situation is much more complicated than that for dimension 8; in fact there are (coincidentally) exactly 16 non-commutative, non-cocommutative (semisimple) Hopf algebras of dimension 16, with exactly seven isomorphism classes of $K_0$-rings.

One might hope that if $K_0(H)$ is commutative, then $H$ is a twist of a group algebra. However this is false; the next example was shown to us by G. Mason and R. Ng.

**Example 5.2.** We consider the Drinfel'd double $D = D(\mathbb{C}S_3)$. Since $D$ is quasi-triangular, $\mathcal{R}(D)$ (and $K_0(D)$) is commutative, as noted in Section 4. However $D$ is not a twist of a group algebra. To see this, we recall that for any finite group $G$, the irreducible representations of $D(\mathbb{C}G)$ arise as follows: choose one element $g$ from each conjugacy class of $G$, consider the centralizer $C(g)$ of $g$ in $G$, and let $V$ be an irreducible representation of $C(g)$. Then the induced module $\mathbb{C}G \otimes_{C(g)} V$ becomes an irreducible $D$-module [Ms], and all irreducible $D$-modules are of this form. Using this one can check that $D(\mathbb{C}S_3)$ has exactly two 1-dimensional irreducible representations. However, there is no group $G$ of order 36 with this property (such a group would have $[G : G'] = 2$, and consequently the Sylow 3-subgroup $P$ of $G'$ is actually normal in $G$. But then $G/P$ is abelian of order 4, a contradiction). Thus $D$ cannot be isomorphic to $\mathbb{C}G$ as an algebra, so is certainly not a twist of it .

If we restrict the situation to twists by cocycles, then much more can be said about equivalence. If $A = B^J$, where $J$ is a cocycle, then the comodule categories are monoidally equivalent. The converse holds when $A$ is finite-dimensional. Thus the three non-commutative dimension 8 Hopf algebras are not (dual) cocycle twists of one another. A result for infinite-dimensional $H$ is proved in [EG4].

The dual situation, at least for ordinary cocycles, has also been studied; see for example [Do][Tk]. We recall that $\sigma : H \otimes H \to k$ is a *Hopf 2-cocycle* if it is convolution invertible, $\sigma(h,1) = \sigma(1,h) = \varepsilon(h)$, and for all $h, l, m \in H$,

$$\sum \sigma(l_1, m_1)\sigma(h, l_2 m_2) = \sigma(h_1, l_1)\sigma(h_2 l_2, m). \tag{5.2}$$

We may then form the twisted Hopf algebra $H_\sigma$. In this case the coalgebra structure is unchanged but the algebra structure is twisted by $\sigma$. That is, $H_\sigma$ has new multiplication

$$h \cdot l := \sum \sigma(h_1, l_1) h_2 l_2 \sigma^{-1}(h_3, l_3),$$

where here we have used the iterated summation notation $(id \otimes \Delta)\Delta(h) = \Sigma h_1 \otimes h_2 \otimes h_3$.

See also [Sca], [Ma4].

## 6. The Schur Indicator

Again we assume $H$ is semisimple over an algebraically closed field of characteristic 0. Just as in the representation theory of finite groups, we may define a *Schur indicator* $\nu(\chi)$ for any irreducible character $\chi$ of $H$. As before choose $\Lambda \in \int_H$ with $\varepsilon(\Lambda) = 1$. Then we define

$$\nu(\chi) := \sum_{(\Lambda)} \chi(\Lambda_1 \Lambda_2).$$

When $H = kG$, let $\Lambda = \frac{1}{|G|}\sum_{g \in G} g$. Then $\nu(\chi) = \frac{1}{|G|}\sum \chi(g^2)$, which is the usual definition for groups [Se].

**Theorem 6.1.** [LM] *Let $H$ be a semisimple Hopf algebra over an algebraically closed field $k$. If $k$ has characteristic $p \neq 0$, assume in addition that $p \neq 2$ and that $H^*$ is semisimple. Then for $\Lambda$ and $\nu(\chi)$ as above, and any irreducible character $\chi \in Irr(H)$, the following properties hold:*

*(1) $\nu(\chi) = 0$, 1 or -1, $\forall \chi \in Irr(H)$,*

*(2) $\nu(\chi) \neq 0$ if and only if $V_\chi \cong V_\chi^*$. Moreover $\nu(\chi) = 1$ (respectively $-1$) if and only if $V_\chi$ admits a symmetric (resp. skew-symmetric) non-degenerate bilinear $H$-invariant form.*

*(3)Considering $S \in End(H)$, $TrS = \sum_{\chi \in Irr(H)} \nu_(\chi)\chi(1_H)$.*

In fact [LM] prove a more general result for arbitrary split semisimple algebras with involution. After stating that result, we will use it to prove Theorem 6.1 in the case of charactersitic 0.

**Theorem 6.2.** *Let $A$ be a finite-dimensional split semisimple algebra over $k$, and write $A = \oplus_{i=1}^d M_{n_i}(k)$. Assume that $k$ has characteristic $\neq 2$ and that each $n_i \neq 0$ in $k$. Assume that $A$ has a $k$-involution $S$. Let $V_1, \ldots, V_d$ be the distinct irreducible modules for $A$ and let $\chi_1, \ldots \chi_d$ be the corresponding irreducible characters. Also let $\{a_r, b_r\}$ be a pair of dual bases with respect to* *some symmetric bilinear associative non-degenerate form $\langle\ |\ \rangle$ on $A$. Then the scalars*

$$\mu(\chi_i) := \frac{n_i}{\chi_i(\sum_j a_j b_j)} \chi_i(\sum_r S(a_r)b_r)$$

*satisfy the follwing properties:*

(1) $\mu(\chi_i) = 0$,*1 or -1, for all* $\chi_i \in Irr(A)$.
(2) $\mu(\chi_i) \neq 0$ *if and only if* $V_i \cong V_i^*$. *Also* $\mu(\chi_i) = 1$ *(respectively -1) if and only if* $V_i$ *admits a symmetric (resp. skew-symmetric) non-degenerate form such that* $S|_{A_i}$ *is the adjoint of the form, where* $A_i$ *is the* $i$*th summand of* $A$.
(3) $Tr_A S = \sum_{\chi \in Irr(A)} \mu(\chi)\chi(1_A)$.

The proof of this theorem involves a careful analysis of various trace functions, in the spirit of some of our previous arguments. In particular it is shown in the proof of Theorem 2.7 that if $\langle a|b\rangle = \sum_i \gamma_i tr_i(ab)$, then $\chi_i(\sum_j a_j b_j) = \gamma_i^{-1} n_i^2$;

*Proof.* (*of Theorem 3.1.*) If $k$ has characteristic 0, then by Theorem 2.1 and 2.2, $S^2 = id$ and both $H$ and $H^*$ are semisimple. Thus $S$ is an involution on $H$ and we may apply Theorem 2.7 to $H$.

We next show that $\mu(\chi) = \nu(\chi)$. Choose $\lambda \in \int_{H^*}$ such that $\lambda(1) = \dim H$ and $\Lambda \in \int_H$ with $\varepsilon(\Lambda) = 1$. Then by Proposition 2.13, $\lambda(\Lambda) = 1$ and

$$\lambda = \sum_{\chi_i \in Irr(H)} n_i \chi_i = \chi_H.$$

Define a bilinear form $\langle\ |\ \rangle$ on $H$ via

$$\langle a|b\rangle = \lambda(ab),$$

for all $a, b \in H$. It is clear that $\langle\ |\ \rangle$ is a non-degenerate associative symmetric bilinear form on $H$, and by Proposition 2.7, $\{S(\Lambda_1), \Lambda_2\}$ is a pair of dual bases with respect to $\langle\ |\ \rangle$.

Now in the bilinear form $\langle\ |\ \rangle$ above, $\gamma_i = n_i$. Thus using the remark before the proof, $\chi_i(\sum_j a_j b_j) = n_i$. Using the dual bases above,

$$\mu(\chi_i) = \chi_i(\sum_j S(a_j)b_j) = \chi_i(\sum_{(\Lambda)} S^2(\Lambda_1)\Lambda_2) = \chi_i(\sum_{(\Lambda)} \Lambda_1\Lambda_2) = \nu(\chi_i),$$

from the definition of $\nu(\chi)$ above.

Thus in order to finish the proof of Theorem 1 we only have to show that if $i = i^*$ then the bilinear form on $V_i$ is $H$-invariant. However this is trivial:

$$\sum_{(h)} (h_1 \cdot v, h_2 \cdot w) = \sum_{(h)} (v, S(h_1)h_2 \cdot w) = \varepsilon(h)(v, w)$$

since $S$ is the adjoint map with respect to the form. $\square$

This invariant can be computed for the representations of various Hopf algebras; in fact Kashina reports that some of the work in [Ksh] could be simplified by using Schur indicators rather than $K_0(H)$ (see [Ksh2]).

As an example, we study in more detail the unique non-commutative, non-cocommutative Hopf algebra $H_8$ of dimension 8 [KacP], mentioned in the last section.

**Example 6.3.** *The Hopf algebra $H_8$*

We give generators and relations for $H = H_8$ as given in [Ma1]:

$$H = k\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy)\rangle$$

The coalgebra structure is given by $x, y \in G(H)$ and $\Delta(z) = \frac{1}{2}[(1+y)\otimes 1 + (1-y)\otimes x](z\otimes z)$. Thus $K := k\langle x, y\rangle \cong k(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $K$ is a normal Hopf subalgebra of $H$. Then $H/HK^+ = k\langle\overline{z}\rangle$, and we see that $\Delta(\overline{z}) = \overline{z}\otimes\overline{z}$ and $\overline{z}^2 = 1$, so that $k\langle\overline{z}\rangle \cong k\mathbb{Z}_2$. Thus we can write $H$ as the extension

$$K \cong k(\mathbb{Z}_2 \times \mathbb{Z}_2) \hookrightarrow H \xrightarrow{\pi} k\mathbb{Z}_2 = k(\overline{z})$$

where $\pi(x) = \pi(y) = 1$ and $\pi(z) = \overline{z}$. For computations, it is easier if we identify $K$ with $K^*$; that is, we may write a basis for $K$ as $\{e_{ij} \mid i, j = 0, 1\}$, where $e_{ij}(x^k y^l) = \delta_{ik}\delta_{jl}$. For example, with this notation,

$$\Delta(z) = (\sum_{i,j,p,r=0}^{1} e_{ij}\otimes e_{pr})(z\otimes z)$$

.

We now compare the Schur indicators of the representations of $H_8$ with that of the other non-commutative Hopf algebras of dimension 8, that is the two group algebras $kD_4$ and $kQ$. Let $\chi$ denote the character of the unique 2-dimensional irreducible representation of each algebra. It is known that for $D_4$, $\nu(\chi) = +1$ and that for $Q$, $\nu(\chi) = -1$.

For $H_8$, an integral is given by $\Lambda = \frac{1}{8}\sum_{ijl=0}^{1} x^i y^j z^l$. Using this, or rather the $\{e_{ij}\}$ above, a direct computation shows that $\nu(\chi) = +1$ (we thank Kashina for the computation). Thus the representation theory of $H_8$ is closer in some sense to that of the dihedral group $D_4$ than to that of $Q$, although their categories of representations are different.

Note that the result of [TY] mentioned in the previous section says that $H_8$ has no non-trivial cocycle (or dual cocycle) deformations.

The Schur indicator is computed for a number of other examples of Hopf algebras, in [KMM]. The examples considered there are all extensions of the dual of a group algebra by another group algebra, as is $H_8$ above.

## References

[CR] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol I, Wiley-Interscience, New York, 1981.

[Do] Y. Doi, Braided bialgebras and quadratic bialgebras, *Comm. Alg* 12 (1993), 1731-1749.

[Dr] V. G. Drinfel'd, On almost cocommutative Hopf algebras, *Leningrad Math J.* 1 (1990), 321-342.

[Dr2] V. G. Drinfel'd, Quasi-Hopf algebras, *Leningrad Math J.* 1 (1990), 1415-1457.

[EG] P. Etingof and S. Gelaki, Some properties of finite-dimensional semisimple Hopf algebras, *Math. Res. Lett.* 5 (1998), 191-197.

[EG2] P. Etingof and S. Gelaki, On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, *Inter. Math. Res. Not.* 16 (1998), 851-864.

[EG3] P. Etingof and S. Gelaki, Semisimple Hopf algebras of dimension $pq$ are trivial, *J. Algebra* 210 (1998), 664-669.

[EG4] P. Etingof and S. Gelaki, On cotriangular Hopf algebras, preprint 1999.

[GW] S. Gelaki and S. Westreich, On semisimple Hopf algebras of dimension $pq$, *Proc. Amer. Math. Soc.* 128 (2000), 2829-2831.

[Kac] G.I. Kac, Certain arithetic properties of ring groups, *Functional Anal. Appl* 6 (1972), 158-160.

[KacP] G. I. Kac and V. G. Paljutkin, Finite ring-groups, *Trudy Moscov. Mat. Ob.* 15 (1966) 224-261 (Russian).

[Kad] L. Kadison, *New Examples of Frobenius Extensions*, University Lecture Notes 14, Amer. Math. Soc., Providence, 1999.

[Kap] I. Kaplansky, *Bialgebras*, University of Chicago Lecture Notes in Math, 1975.

[Ksh] Y. Kashina, Classification of semisimple Hopf algebras of dimension 16, *J. of Algebra* 232 (2000), 617-663.

[Ksh2] Y. Kashina, On semisimple Hopf algebras of dimension $2^n$, *Algebras and Representation Theory*, to appear.

[KMM] Y. Kashina, G. Mason, and S. Montgomery, Computing the Schur indicator for some finite-dimensional Hopf algebras, in preparation.

[Ksl] C. Kassell, *Quantum Groups*, GTM 155, Springer-Verlag, New York-Berlin, 1995.

[KM] T. Kobayashi and A. Masuoka, A result extended from groups to Hopf algebras, *Tsukuba J. Math* 21 (1997), 55-58.

[L] R. G. Larson, Characters of Hopf algebras, *J. of Algebra* 17 (1971), 352-368.

[LR1] R. Larson and D. Radford, Semisimple cosemisimple Hopf algebras, *Amer. J. Math* 109 (1987), 287-295.

[LR2] R. Larson and D. Radford, Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra* 117 (1988), 267-289.

[LS] R. Larson and M. E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math* 91 (1969), 75-93.

[LM] V. Linchenko and S. Montgomery, A Frobenius-Schur theorem for Hopf algebras, *Algebras and Representation Theory* 3 (2000), 347-355.

[Lo] M. Lorenz, On the class equation for Hopf algebras, *Proc. Amer. Math. Soc.* 126 (1998), 2841-2844.

[Ms] G. Mason, The quantum double of a finite group and its role in conformal field theory, Groups '93 Galway/St. Andrews, Vol 2, 405-417, *LMS Lecture Notes* 212, Cambridge University Press, Cambridge, 1995.

[Ma1] A. Masuoka, Semisimple Hopf algebras of dimension 6, 8, *Israel J. Math.* 92 (1995), 361-373.

[Ma2] A. Masuoka, The $p^n$ theorem for semisimple Hopf algebras, *Proc. Amer. Math. Soc.* 124 (1996), 735-737.

[Ma3] A. Masuoka, Self-dual Hopf algebras of dimension $p^3$ obtained by extension, *J. Algebra* 178 (1995), 791-806.

[Ma4] A. Masuoka, Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension, *AMS Contemporary Math*, to appear.

[M] S. Montgomery, *Hopf Algebras and their Actions on Rings*, CBMS Lectures, Vol. 82, AMS, Providence, RI, 1993.

[MW] S. Montgomery and S. Witherspoon, Irreducible Representations of Crossed Products, *J. Pure and Applied Algebra* 129 (1998), 315-326.

[Na] S. Natale, On semisimple Hopf algebras of dimension $pq^2$, *J. Algebra* 221 (1999), 242-278.

[NR] W. Nichols and M. B. Richmond, The Grothendieck group of a Hopf algebra, *J. Pure and Appl. Algebra*, 106 (1996), 297-306.

[N] D. Nikshych, $K_0$-rings and twisting of finite dimensional semisimple Hopf algebras, *Comm. Algebra* 26 (1998) 321-342; Erratum: *Comm. Algebra* 26 (1998) 1347.

[OS] U. Oberst and H.-J. Schneider, Über Untergruppen endlicher algebraischer Gruppen, *Manuscripta Math.* 8 (1973), 217-241.

[R] D. Radford, The order of the antipode of a finite-dimensional Hopf algebra is finite, *Amer. J. Math* 98 (1976), 333-355.

[R2] D. Radford, Minimal quasi-triangular Hopf algebras, *J. Algebra* 157 (1993), 285-315.

[Re] N. Reshetikhin, Multiparameter quantum groups and twisted quasi-Hopf algebras, *Letters Math.Physics* 20 (1990), 331=335.

[Sca] P. Schauenburg, Hopf bigalois extensions, *Comm. Algebra* 24 (1996), 3797-3825.

[Sch1] H.-J. Schenider, *Lectures on Hopf algebras*, Trabajos Mat. 31/95, IMAF, Universidad Nac. Cordoba, Argentina, 1995. Available at www.mate.uncor.edu/andrus.

[Sch2] H.-J. Schneider, Some properties of factorizable Hopf algebras, *Proc. Amer. Math. Soc.*, to appear.

[Se] J. P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, Berlin and New York, 1977.

[TY] D. Tambara and S. Yamagami, Tensor categories with fusion rules of self-duality for abelian groups, *J. Algebra* 209 (1998), 692-707.

[Tk] M. Takeuchi, Cocycle deformations of coordinate rings of quantum matrices, *J. Algebra* 189 (1997), 23-33.

[V] L. Vainerman, 2-cocycles and twisting of Kac algebras, *Comm. Math Physics* 191 (1998), 697-721.

[Z] Y. Zhu, Hopf algebras of prime dimension, *Internat. Math. Res. Not.* (1994), 53-59.

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