Two-sided (two-cosided) Hopf modules and Doi-Hopf modules for quasi-Hopf algebras*

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Abstract

Let H be a finite dimensional quasi-Hopf algebra over a field k and $\mathfrak{A}$ a right H-comodule algebra in the sense of [12]. We first show that on the k-vector space $\mathfrak{A}\otimes H^*$ we can define an algebra structure, denoted by $\mathfrak{A}\,\overline{\#}\,H^*$, in the monoidal category of left H-modules (i.e. $\mathfrak{A}\,\overline{\#}\,H^*$ is an H-module algebra in the sense of [2]). Then we will prove that the category of two-sided $(\mathfrak{A},H)$-bimodules ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of relative $(\mathfrak{A}\,\overline{\#}\,H^*,H^*)$-Hopf modules, as introduced in [3]. In the particular case where $\mathfrak{A}=H$, we will obtain the Nill's result announced in [14]. We will also introduce the categories of Doi-Hopf modules and two-sided two-cosided Hopf modules and we will show that they are in certain situations isomorphic to module categories.

0 Introduction

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [11] in connection with the Knizhnik-Zamolodchikov equations [15]. Let k be a field, H an associative algebra and $\Delta:\ H\to H\otimes H$ and $\varepsilon:\ H\to k$ two algebra morphisms. Roughly speaking, H is a quasi-bialgebra if the category ${}_H\mathcal{M}$ of left H-modules, equipped with the tensor product of vector spaces endowed with the diagonal H-module structure given via Δ, and with unit object k viewed as a left H-module via ε, is a monoidal category. The comultiplication Δ is not coassociative but is quasi-coassociative in the sense that Δ is coassociative up to conjugation by an invertible element $\Phi\in H\otimes H\otimes H$. Moreover, H is a quasi-Hopf algebra if and only if each finite dimensional left H-module has a dual H-module. Note that, the definition of a quasi-bialgebra and a quasi-Hopf algebra is not self dual.

Since H is not a coassociative coalgebra, it is impossible to define comodules over H. However, since H is a coalgebra in the monoidal category of (H,H)-bimodules ${}_H\mathcal{M}_H$, we can define Hopf modules in ${}_H\mathcal{M}_H^H$. More exactly, a Hopf module in ${}_H\mathcal{M}_H^H$ is a right H-comodule in the monoidal category ${}_H\mathcal{M}_H$, cf. [14]. Also, we can define a Hopf module in ${}_H^H\mathcal{M}_H^H$: a Hopf module in ${}_H^H\mathcal{M}_H^H$ is an H-H-bicomodule in the monoidal category ${}_H\mathcal{M}_H$, cf. [19].

Using the theory of (co)algebras and (co)modules in monoidal categories we can define categories of relative Hopf modules. If H is a finite dimensional quasi-bialgebra and A an algebra in the monoidal

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category ${}_H\mathcal{M}$, then a relative Hopf module in $\mathcal{M}_A^{H^*}$ is a right H^*-comodule (H^* is a coassociative coalgebra) which is also a right A-module in the monoidal category $\mathcal{M}^{H^*}$, see [3]. Similarly, if we start with a coalgebra C in the monoidal category of right H-modules $\mathcal{M}_H$, then a relative Hopf module in ${}^C\mathcal{M}_H$ is a left C-comodule in the monoidal category $\mathcal{M}_H$. Of course, when H is an ordinary bialgebra these categories are exactly the ones defined by Doi in [8]. Unfortunately, at the moment, the basic category of Hopf modules $\mathcal{M}_H^H$ is not defined (a possible way to define this category will be presented later in Section 3).

In this paper, our goal is to generalize all the categories presented above and to prove that, in the finite dimensional case, they are isomorphic to module categories. Essential tools will be the notions of comodule algebra and bicomodule algebra over a quasi-bialgebra H [12]. An associative algebra $\mathfrak{A}$ is called a right H-comodule algebra if there exists an algebra map $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ which is "almost" associative, in a way similar to the "almost" coassociativity of the comultiplication on a quasi-bialgebra; a detailed definition will be presented below. In a similarl way, we can define the notion of left H-comodule algebra. If H is a finite dimensional quasi-bialgebra and $\mathfrak{A}$ a right H-comodule algebra, then we can define a multiplication on the k-vector space $\mathfrak{A} \otimes H^*$; we obtain an algebra in the monoidal category ${}_H\mathcal{M}$ (i.e. a left H-module algebra), which will be denoted by $\mathfrak{A} \,\overline{\#}\, H^*$. Notice that, in the Hopf case, $\mathfrak{A} \,\overline{\#}\, H^*$ is just the usual smash product of $\mathfrak{A}$ and H^*, and this is why we call the monoidal algebra $\mathfrak{A} \,\overline{\#}\, H^*$ the quasi-smash product of $\mathfrak{A}$ and H^*. Also remark that, in the quasi-Hopf case, $\mathfrak{A} \,\overline{\#}\, H^*$ is not an associative algebra because the associativity constraints of ${}_H\mathcal{M}$ are not the trivial ones but the ones given by the reassociator Φ of H.

In Section 3 we will define the category of two-sided $(\mathfrak{A}, H)$-Hopf modules ${}_H\mathcal{M}_{\mathfrak{A}}^H$. Concerning its description in terms of relative Hopf modules, a central role is played by $\mathfrak{A} \,\overline{\#}\, H^*$. First, the definition of a two-sided $(\mathfrak{A}, H)$-Hopf module is slightly more general than the one given for a Hopf module in ${}_H\mathcal{M}_H^H$. Its objects are H-$\mathfrak{A}$-bimodules and "almost" right H-comodules such that, in the usual way, the right H-coaction is left H-linear an a right $\mathfrak{A}$-linear map. If $\mathfrak{A} = H$, then ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is just the category of Hopf modules ${}_H\mathcal{M}_H^H$ described above. Secondly, if H is a finite dimensional quasi-Hopf algebra then ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of relative $(\mathfrak{A} \,\overline{\#}\, H^*, H^*)$-Hopf modules (this generalizes [7, Proposition 2.3]). Following [3], we can show in this case that ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of right $(\mathfrak{A} \,\overline{\#}\, H^*)\#H$-modules, where $(\mathfrak{A} \,\overline{\#}\, H^*)\#H$ denotes the smash product algebra (in the sense of [2]) of $\mathfrak{A} \,\overline{\#}\, H^*$ and H. In particular, taking $\mathfrak{A} = H$ we recover Nill's result, as announced in [14], which states that ${}_H\mathcal{M}_H^H$ is isomorphic to the category of right modules over the two-sided crossed product $H \rtimes H^* \ltimes H$. In Section 4 we will prove that the two-sided crossed product constructed in [12] is in fact a generalized smash product. As a consequence, $(H \,\overline{\#}\, H^*)\#H$ is just the two-sided crossed product $H \rtimes H^* \ltimes H$ (as an algebra).

In the second part of the paper we will study the category of two-sided two-cosided Hopf modules ${}_H^C\mathcal{M}_{\mathbb{A}}^H$. Here C is a coalgebra in the monoidal category of (H, H)-bimodules ${}_H\mathcal{M}_H$ (i.e. an H-bimodule coalgebra), and $\mathbb{A}$ is an H-bicomodule algebra in the sense of [12]. Roughly speaking, an object in ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is a two-sided $(\mathbb{A}, H)$-Hopf module which is also an "almost" left C-comodule such that the left C-coaction is compatible with the other structure maps. In Section 5 we will show that if C and H are finite dimensional then ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to a category of right modules. To this end we will describe first ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ as a category of Doi-Hopf modules. If $\mathfrak{B}$ is a left H-comodule algebra and C is a right H-module coalgebra then the category of right-left $(H, \mathfrak{B}, C)$-Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a straightforward generalization of the category of relative Hopf modules ${}^C\mathcal{M}_H$. When C is finite dimensional, ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{B}$. Now, returning to the category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$, if H is finite dimensional then we will show that $(\mathbb{A} \,\overline{\#}\, H^*)\#H$ is a left $H \otimes H^{\mathrm{op}}$-comodule algebra (here "op" means the opposite multiplication on H) so, it makes sense to consider the category of Doi-Hopf modules ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A} \,\overline{\#}\, H^*)\#H}$. Moreover, the main results

asserts that ${}^{C}_{H}\mathcal{M}^{H}_{\mathbb{A}}$ is isomorphic to ${}^{C}\mathcal{M}(H\otimes H^{\mathrm{op}})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$ (this generalizes [1, Proposition 2.3]). In particular, if C is also finite dimensional, we obtain that ${}^{C}_{H}\mathcal{M}^{H}_{\mathbb{A}}$ is isomorphic to the category of right modules over the generalized smash product $\mathcal{A}=C^*\blacktriangleright\!\!<((\mathbb{A}\,\overline{\#}\,H^*)\#H)$. In the Hopf case, the left-handed version of this result was first obtained by Cibils and Rosso [6]. More precisely, they define an algebra X having the property that the category ${}^{H^*}_{H^*}\mathcal{M}^{H^*}_{H^*}$ is isomorphic to the category of left X-modules. Recently, Panaite [18] introduced two other algebras Y and Z with the same property as X. Y is the two-sided crossed product $H^*\#(H\otimes H^{\mathrm{op}})\#H^{*\mathrm{op}}$ and Z is the diagonal crossed product (in the sense of [12]) $(H^*\otimes H^{*\mathrm{op}})\bowtie(H\otimes H^{\mathrm{op}})$.

1 Preliminary results

Quasi-Hopf algebras

We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [11], a quasi-bialgebra is a fourtuple $(H,\Delta,\varepsilon,\Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H\otimes H\otimes H$, and $\Delta:\ H\to H\otimes H$ and $\varepsilon:\ H\to k$ are algebra homomorphisms satisfying the identities

$$(id\otimes\Delta)(\Delta(h))=\Phi(\Delta\otimes id)(\Delta(h))\Phi^{-1}, \tag{1.1}$$

$$(id\otimes\varepsilon)(\Delta(h))=h\otimes 1,\quad (\varepsilon\otimes id)(\Delta(h))=1\otimes h, \tag{1.2}$$

for all $h\in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$(1\otimes\Phi)(id\otimes\Delta\otimes id)(\Phi)(\Phi\otimes 1)=(id\otimes id\otimes\Delta)(\Phi)(\Delta\otimes id\otimes id)(\Phi), \tag{1.3}$$

$$(id\otimes\varepsilon\otimes id)(\Phi)=1\otimes 1\otimes 1. \tag{1.4}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras [21] we denote $\Delta(h)=\sum h_1\otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention

$$(\Delta\otimes id)(\Delta(h))=\sum h_{(1,1)}\otimes h_{(1,2)}\otimes h_2,\quad (id\otimes\Delta)(\Delta(h))=\sum h_1\otimes h_{(2,1)}\otimes h_{(2,2)},$$

for all $h\in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi=\sum X^1\otimes X^2\otimes X^3=\sum T^1\otimes T^2\otimes T^3=\sum V^1\otimes V^2\otimes V^3=\cdots$$
$$\Phi^{-1}=\sum x^1\otimes x^2\otimes x^3=\sum t^1\otimes t^2\otimes t^3=\sum v^1\otimes v^2\otimes v^3=\cdots$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-automorphism S of the algebra H and elements $\alpha,\beta\in H$ such that, for all $h\in H$, we have:

$$\sum S(h_1)\alpha h_2=\varepsilon(h)\alpha\quad\text{and}\quad\sum h_1\beta S(h_2)=\varepsilon(h)\beta, \tag{1.5}$$

$$\sum X^1\beta S(X^2)\alpha X^3=1\quad\text{and}\quad\sum S(x^1)\alpha x^2\beta S(x^3)=1. \tag{1.6}$$

For a quasi-Hopf algebra the antipode is determined uniquely up to a transformation $\alpha\mapsto U\alpha$, $\beta\mapsto\beta U^{-1}$, $S(h)\mapsto US(h)U^{-1}$, where $U\in H$ is invertible. The axioms for a quasi-Hopf algebra imply that

$\varepsilon(\alpha)\varepsilon(\beta)=1$, so, by rescaling α and β, we may assume without loss of generality that $\varepsilon(\alpha)=\varepsilon(\beta)=1$ and $\varepsilon\circ S=\varepsilon$. The identities (1.2), (1.3) and (1.4) also imply that

$$(\varepsilon\otimes id\otimes id)(\Phi)=(id\otimes id\otimes\varepsilon)(\Phi)=1\otimes 1\otimes 1. \tag{1.7}$$

Next we recall that the definition of a quasi-Hopf algebra is "twist coinvariant" in the following sense. An invertible element $F\in H\otimes H$ is called a *gauge transformation* or *twist* if $(\varepsilon\otimes id)(F)=(id\otimes\varepsilon)(F)=1$. If H is a quasi-Hopf algebra and $F=\sum F^1\otimes F^2\in H\otimes H$ is a gauge transformation with inverse $F^{-1}=\sum G^1\otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, antipode and the elements α and β by

$$\Delta_F(h)=F\Delta(h)F^{-1}, \tag{1.8}$$

$$\Phi_F=(1\otimes F)(id\otimes\Delta)(F)\Phi(\Delta\otimes id)(F^{-1})(F^{-1}\otimes 1), \tag{1.9}$$

$$\alpha_F=\sum S(G^1)\alpha G^2,\quad \beta_F=\sum F^1\beta S(F^2). \tag{1.10}$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f\in H\otimes H$ such that

$$f\Delta(S(h))f^{-1}=\sum(S\otimes S)(\Delta^{\mathrm{op}}(h)),\quad \text{for all } h\in H, \tag{1.11}$$

where $\Delta^{\mathrm{op}}(h)=\sum h_2\otimes h_1$. f can be computed explicitly. First set

$$\sum A^1\otimes A^2\otimes A^3\otimes A^4=(\Phi\otimes 1)(\Delta\otimes id\otimes id)(\Phi^{-1}), \tag{1.12}$$

$$\sum B^1\otimes B^2\otimes B^3\otimes B^4=(\Delta\otimes id\otimes id)(\Phi)(\Phi^{-1}\otimes 1) \tag{1.13}$$

and then define $\gamma,\delta\in H\otimes H$ by

$$\gamma=\sum S(A^2)\alpha A^3\otimes S(A^1)\alpha A^4\ \text{ and }\ \delta=\sum B^1\beta S(B^4)\otimes B^2\beta S(B^3). \tag{1.14}$$

f and f^{-1} are then given by the formulas

$$f = \sum(S\otimes S)(\Delta^{\mathrm{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \tag{1.15}$$

$$f^{-1} = \sum\Delta(S(x^1)\alpha x^2)\delta(S\otimes S)(\Delta^{\mathrm{op}}(x^3)). \tag{1.16}$$

f satisfies the following relations:

$$f\Delta(\alpha)=\gamma,\quad \Delta(\beta)f^{-1}=\delta. \tag{1.17}$$

Furthermore the corresponding twisted reassociator (see (1.9)) is given by

$$\Phi_f=\sum(S\otimes S\otimes S)(X^3\otimes X^2\otimes X^1). \tag{1.18}$$

In a Hopf algebra H, we obviously have the identity

$$\sum h_1\otimes h_2S(h_3)=h\otimes 1,\ \text{for all } h\in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [12], [13], we define

$$p_R=\sum p_R^1\otimes p_R^2=\sum x^1\otimes x^2\beta S(x^3),\quad q_R=\sum q_R^1\otimes q_R^2=\sum X^1\otimes S^{-1}(\alpha X^3)X^2, \tag{1.19}$$

$$p_L = \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1\beta) \otimes X^3, \quad q_L = \sum q_L^1 \otimes q_L^2 = \sum S(x^1)\alpha x^2 \otimes x^3. \tag{1.20}$$

For all $h \in H$, we then have

$$\sum \Delta(h_1) p_R [1 \otimes S(h_2)] = p_R[h \otimes 1], \quad \sum [1 \otimes S^{-1}(h_2)] q_R \Delta(h_1) = (h \otimes 1) q_R, \tag{1.21}$$

$$\sum \Delta(h_2) p_L [S^{-1}(h_1) \otimes 1] = p_L(1 \otimes h), \quad \sum [S(h_1) \otimes 1] q_L \Delta(h_2) = (1 \otimes h) q_L, \tag{1.22}$$

and

$$\sum \Delta(q_R^1) p_R [1 \otimes S(q_R^2)] = 1 \otimes 1, \quad \sum [1 \otimes S^{-1}(p_R^2)] q_R \Delta(p_R^1) = 1 \otimes 1, \tag{1.23}$$

$$\sum [S(p_L^1) \otimes 1] q_L \Delta(p_L^2) = 1 \otimes 1, \quad \sum \Delta(q_L^2) p_L [S^{-1}(q_L^1) \otimes 1] = 1 \otimes 1, \tag{1.24}$$

$$\begin{aligned} &(q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} \\ &\quad = \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)][1 \otimes S^{-1}(f^2) \otimes S^{-1}(f^1)](id \otimes \Delta)(q_R \Delta(X^1)), \end{aligned} \tag{1.25}$$

$$\begin{aligned} &\Phi(\Delta \otimes id)(p_R)(p_R \otimes id) \\ &\quad = \sum (id \otimes \Delta)(\Delta(x^1) p_R)(1 \otimes f^{-1})(1 \otimes S(x^3) \otimes S(x^2)), \end{aligned} \tag{1.26}$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (1.15).

The smash product

Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left (right) H-modules, define $a_{U,V,W}, \mathbf{a}_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)),$$
$$\mathbf{a}_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi^{-1}.$$

Then the category ${}_H\mathcal{M}$ ($\mathcal{M}_H$) of left (right) H-modules becomes a monoidal category (see [15], [17] for the terminology) with tensor product $\otimes$ given via Δ, associativity constraints $a_{U,V,W}$ ($\mathbf{a}_{U,V,W}$), unit k as a trivial H-module and the usual left and right unit constraints.
Now, let H be a quasi-bialgebra. We say that a k-vector space A is a left H-module algebra if it is an algebra in the monoidal category ${}_H\mathcal{M}$, that is A has a multiplication and a usual unit 1_A satisfying the following conditions:

$$(aa')a'' = \sum (X^1 \cdot a)[(X^2 \cdot a')(X^3 \cdot a'')], \tag{1.27}$$

$$h \cdot (aa') = \sum (h_1 \cdot a)(h_2 \cdot a'), \tag{1.28}$$

$$h \cdot 1_A = \varepsilon(h) 1_A \tag{1.29}$$

for all $a, a', a'' \in A$ and $h \in H$, where $h \otimes a \to h \cdot a$ is the H-module structure of A. Following [2] we define the smash product $A\#H$ as follows: as a vector space $A\#H$ is $A \otimes H$ ($a \otimes h$ viewed as an element of $A\#H$ will be written $a\#h$) with multiplication given by

$$(a\#h)(a'\#h') = \sum (x^1 \cdot a)(x^2 h_1 \cdot a')\#x^3 h_2 h', \tag{1.30}$$

for all $a, a' \in A$, $h, h' \in H$. $A\#H$ is an associative algebra and it is defined by a universal property (as Heyneman and Sweedler did for Hopf algebras, see [2]). It is easy to see that H is a subalgebra of $A\#H$ via $h \mapsto 1\#h$, A is a k-subspace of $A\#H$ via $a \mapsto a\#1$ and the following relations hold:

$$(a\#h)(1\#h') = a\#hh', \quad (1\#h)(a\#h') = \sum h_1 \cdot a\#h_2 h', \tag{1.31}$$

for all $a \in A$, $h, h' \in H$.
For further use we need also the notion of right H-module coalgebra. Suppose that H is a quasi-bialgebra. Since the category of right H-modules is a monoidal category we can define coalgebras in this category. So, we say that a k-linear space C is a right H-module coalgebra if C is a coalgebra in the monoidal category $\mathcal{M}_H$, that is, if C is a right H-module (denote by $c \cdot h$ the action of h on c) and has a comultiplication $\underline{\Delta}: C \to C \otimes C$ and a usual counit $\underline{\varepsilon}: C \to k$ satisfying the following relations

$$(\underline{\Delta} \otimes id_C)(\underline{\Delta}(c))\Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)) \ \ \forall\, c \in C, \tag{1.32}$$

$$\underline{\Delta}(c \cdot h) = \sum c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2 \ \ \forall\, c \in C,\, h \in H, \tag{1.33}$$

$$\underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h) \ \ \forall\, c \in C,\, h \in H \tag{1.34}$$

where we use the Sweedler-type notation

$$\underline{\Delta}(c) = c_{\underline{1}} \otimes c_{\underline{2}}, \ \ (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = \sum c_{(\underline{1},\underline{1})} \otimes c_{(\underline{1},\underline{2})} \otimes c_{\underline{2}} \ \ \text{etc.}$$

2 The quasi-smash product

Recall from [12] the notion of comodule algebra over a quasi-bialgebra.

Definition 2.1 *Let H be a quasi-bialgebra. A unital associative algebra $\mathfrak{A}$ is called a right H-comodule algebra if there exist an algebra morphism $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that*

$$\Phi_\rho(\rho \otimes id)(\rho(\mathfrak{a})) = (id \otimes \Delta)(\rho(\mathfrak{a}))\Phi_\rho \ \ \forall\, \mathfrak{a} \in \mathfrak{A}, \tag{2.1}$$

$$(1_{\mathfrak{A}} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho), \tag{2.2}$$

$$(id \otimes \varepsilon) \circ \rho = id, \tag{2.3}$$

$$(id \otimes \varepsilon \otimes id)(\Phi_\rho) = (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H. \tag{2.4}$$

Similarly, a unital associative algebra $\mathfrak{B}$ is called a left H-comodule algebra if there exist an algebra morphism $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$ such that the following relations hold

$$(id \otimes \lambda)(\lambda(\mathfrak{b}))\Phi_\lambda = \Phi_\lambda(\Delta \otimes id)(\lambda(\mathfrak{b})) \ \ \forall\, \mathfrak{b} \in \mathfrak{B}, \tag{2.5}$$

$$(1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes id)(\Phi_\lambda)(\Phi \otimes 1_{\mathfrak{B}}) = (id \otimes id \otimes \lambda)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_\lambda), \tag{2.6}$$

$$(\varepsilon \otimes id) \circ \lambda = id, \tag{2.7}$$

$$(id \otimes \varepsilon \otimes id)(\Phi_\lambda) = (\varepsilon \otimes id \otimes id)(\Phi_\lambda) = 1_H \otimes 1_{\mathfrak{B}}. \tag{2.8}$$

When H is a quasi-bialgebra, particular examples of left and right H-comodule algebras are given by $\mathfrak{A} = \mathfrak{B} = H$ and $\rho = \lambda = \Delta$, $\Phi_\rho = \Phi_\lambda = \Phi$.
For a right H-comodule algebra $(\mathfrak{A}, \rho, \Phi_\rho)$ we will denote

$$\rho(\mathfrak{a}) = \sum \mathfrak{a}_{<0>} \otimes \mathfrak{a}_{<1>}, \ \ (\rho \otimes id)(\rho(\mathfrak{a})) = \sum \mathfrak{a}_{<0,0>} \otimes \mathfrak{a}_{<0,1>} \otimes \mathfrak{a}_{<1>} \ \ \text{etc.}$$

for any $\mathfrak{a} \in \mathfrak{A}$. Similarly, for a left H-comodule algebra $(\mathfrak{B}, \lambda, \Phi_\lambda)$, if $\mathfrak{b} \in \mathfrak{B}$ then we will denote

$$\lambda(\mathfrak{b}) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[1]}, \ \ (id \otimes \lambda)(\lambda(\mathfrak{b})) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0,-1]} \otimes \mathfrak{b}_{[0,0]} \ \ \text{etc.}$$

In analogy with the notation of the reassociator Φ of H, we will write

$$\Phi_\rho = \sum \tilde{X}^1_\rho \otimes \tilde{X}^2_\rho \otimes \tilde{X}^3_\rho = \sum \tilde{Y}^1_\rho \otimes \tilde{Y}^2_\rho \otimes \tilde{Y}^3_\rho = \text{ etc.}$$

and its inverse by

$$\Phi^{-1}_\rho = \sum \tilde{x}^1_\rho \otimes \tilde{x}^2_\rho \otimes \tilde{x}^3_\rho = \sum \tilde{y}^1_\rho \otimes \tilde{y}^2_\rho \otimes \tilde{y}^3_\rho = \text{ etc.}$$

Similarly for the element Φ_λ of a left H-comodule algebra $\mathfrak{B}$. If there is no danger of confusion we will omit the subscription ρ or λ for the tensor components of the elements Φ_ρ, Φ_λ or for the tensor components of the elements Φ^{-1}_ρ, Φ^{-1}_λ.

Suppose now that H is finite dimensional and $\mathfrak{A}$ is a right H-comodule algebra. In the Hopf case, we have that $\mathfrak{A}$ is a left H^*-module algebra so we can consider the smash product $\mathfrak{A}\#H^*$. We will see that a similar result holds in the quasi-Hopf case. Since the resulting object of $\mathfrak{A}$ and H^* is an algebra in the monoidal category ${}_H\mathcal{M}$ (i.e. an H-module algebra) we will call it the quasi-smash product between $\mathfrak{A}$ and H^*. As expected, when H is a bialgebra the two constructions, the smash product and the quasi-smash product, coincide.

Let $\{e_i\}_{i=\overline{1,n}}$ be a basis of H, and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*. H^* is a coassociative coalgebra, with comultiplication

$$\widehat{\Delta}(\varphi) = \sum \varphi_1 \otimes \varphi_2 = \sum_{i,j=1}^{n} \varphi(e_i e_j) e^i \otimes e^j,$$

or, equivalently,

$$\widehat{\Delta}(\varphi) = \sum \varphi_1 \otimes \varphi_2 \iff \varphi(hh') = \sum \varphi_1(h)\varphi_2(h'), \quad \forall h, h' \in H.$$

H^* is also an (H,H)-bimodule, by

$$< h \rightharpoonup \varphi, h' >= \varphi(h'h), \quad < \varphi \leftharpoonup h, h' >= \varphi(hh').$$

The convolution $< \varphi\psi, h >= \sum \varphi(h_1)\psi(h_2)$, $h \in H$, is a multiplication on H^*; it is not associative, but only quasi-associative:

$$[\varphi\psi]\xi = \sum (X^1 \rightharpoonup \varphi \leftharpoonup x^1)[(X^2 \rightharpoonup \psi \leftharpoonup x^2)(X^3 \rightharpoonup \xi \leftharpoonup x^3)], \quad \forall \varphi, \psi, \xi \in H^*. \tag{2.9}$$

In addition, for all $h \in H$ and $\varphi, \psi \in H^*$ we have that

$$h \rightharpoonup (\varphi\psi) = \sum (h_1 \rightharpoonup \varphi)(h_2 \rightharpoonup \psi) \quad \text{and} \quad (\varphi\psi) \leftharpoonup h = \sum (\varphi \leftharpoonup h_1)(\psi \leftharpoonup h_2). \tag{2.10}$$

In other words, H^* is an algebra in the monoidal category of (H,H)-bimodules ${}_H\mathcal{M}_H$.

Now, if $(\mathfrak{A}, \rho, \Psi_\rho)$ is a right H-comodule algebra we define a multiplication on $\mathfrak{A} \otimes H^*$ as follows

$$(\mathfrak{a} \,\overline{\#}\, \varphi)(\mathfrak{a}' \,\overline{\#}\, \Psi) = \sum \mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1 \,\overline{\#}\, (\varphi \leftharpoonup \mathfrak{a}'_{<1>} \tilde{x}^2)(\psi \leftharpoonup \tilde{x}^3) \quad \forall \mathfrak{a}, \mathfrak{a}' \in \mathfrak{A} \text{ and } \varphi, \Psi \in H^* \tag{2.11}$$

where we write $\mathfrak{a} \,\overline{\#}\, \varphi$ for $\mathfrak{a} \otimes \varphi$, $\rho(\mathfrak{a}) = \sum \mathfrak{a}_{<0>} \otimes \mathfrak{a}_{<1>}$, and $\Phi^{-1}_\rho = \sum \tilde{x}^1 \otimes \tilde{x}^2 \otimes \tilde{x}^3$. We denote this structure on $\mathfrak{A} \otimes H^*$ by $\mathfrak{A} \,\overline{\#}\, H^*$.

Proposition 2.2 *Let H be a finite dimensional quasi-bialgebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H-comodule algebra. Then $\mathfrak{A} \,\overline{\#}\, H^*$ is a H-module algebra with unit $1_{\mathfrak{A}} \,\overline{\#}\, \varepsilon$ and with the left H-action given by*

$$h \cdot (\mathfrak{a} \,\overline{\#}\, \varphi) = \mathfrak{a} \,\overline{\#}\, h \rightharpoonup \varphi, \quad \forall\, h \in H,\ \mathfrak{a} \in \mathfrak{A},\ \textit{and}\ \varphi \in H^*. \tag{2.12}$$

Proof. Since H^* is a left H-module via the action $\rightharpoonup$, it is easy to see that $\mathfrak{A}\,\overline{\#}\,H^*$ is a left H-module via the action (2.12). Now, we will prove that $\mathfrak{A}\,\overline{\#}\,H^*$ is an algebra in ${}_H\mathcal{M}$ with unit $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$. Indeed, for all $\mathfrak{a},\mathfrak{a}',\mathfrak{a}''\in\mathfrak{A}$ and $\varphi,\psi,\chi\in H^*$

$$\begin{aligned}
&[X^1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)]\{[X^2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)][X^3\cdot(\mathfrak{a}''\,\overline{\#}\,\chi)]\}\\
&\quad= \sum(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi)[(\mathfrak{a}'\,\overline{\#}\,X^2\rightharpoonup\psi)(\mathfrak{a}''\,\overline{\#}\,X^3\rightharpoonup\chi)]\\
&\quad= \sum(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi)[\mathfrak{a}'\mathfrak{a}''_{<0>}\tilde{x}^1\,\overline{\#}\,(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^2)(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3)]\\
(2.10)\quad&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\mathfrak{a}''_{<0,0>}\tilde{x}^1_{<0>}\tilde{y}^1\,\overline{\#}\,(X^1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\mathfrak{a}''_{<0,1>}\tilde{x}^1_{<1>}\tilde{y}^2)\\
&\qquad[(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^2\tilde{y}^3_1)(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3\tilde{y}^3_2)]\\
(2.9,2.2)\quad&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\mathfrak{a}''_{<0,0>}\tilde{x}^1\tilde{y}^1\,\overline{\#}\,[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\mathfrak{a}''_{<0,1>}\tilde{x}^2\tilde{y}^2_1)(\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^3\tilde{y}^2_2)]\\
&\qquad(\chi\leftharpoonup\tilde{y}^3)\\
(2.1,2.10)\quad&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\mathfrak{a}''_{<0>}\tilde{y}^1\,\overline{\#}\,\{[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)]\leftharpoonup\mathfrak{a}''_{<1>}\tilde{y}^2\}(\chi\leftharpoonup\tilde{y}^3)\\
&= \sum[\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)](\mathfrak{a}''\,\overline{\#}\,\chi)\\
&= [(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)](\mathfrak{a}''\,\overline{\#}\,\chi).
\end{aligned}$$

It is not hard to see that $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ is the unit of $\mathfrak{A}\,\overline{\#}\,H^*$ and that $h\cdot(1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon)=\varepsilon(h)1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ for all $h\in H$. Finally, for all $h\in H$, $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$ and $\varphi,\psi\in H^*$, we calculate:

$$\begin{aligned}
\sum[h_1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)][h_2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)]&=\sum(\mathfrak{a}\,\overline{\#}\,h_1\rightharpoonup\varphi)(\mathfrak{a}'\,\overline{\#}\,h_2\rightharpoonup\psi)\\
&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,(h_1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(h_2\rightharpoonup\psi\leftharpoonup\tilde{x}^3)\\
(2.10)\quad&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,h\rightharpoonup[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)]\\
(2.12)\quad&= h\cdot[(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)].
\end{aligned}$$

□

(H,Δ,Φ) is a right H-comodule algebra, so it makes sense to consider the quasi-smash product $H\,\overline{\#}\,H^*$. In this case where H is a Hopf algebra, $H\#H^*$ is called the Heisenberg double of H, and we will keep the same terminology for quasi-Hopf algebras. $\mathcal{H}(H)=H\,\overline{\#}\,H^*$ is not an associative algebra but it is an algebra in the monoidal category ${}_H\mathcal{M}$. If H is a finite dimensional Hopf algebra then $\mathcal{H}(H)$ is isomorphic to the algebra $\mathrm{End_k(H)}$. In order to prove a similar result for a finite dimensional quasi-Hopf algebra, we first have to deform the algebra structure of $\mathrm{End_k(H)}$.

Proposition 2.3 *Let H be a finite dimensional quasi-Hopf algebra. Define*

$$\mu:\ H\,\overline{\#}\,H^*\to\mathrm{End_k(H)},\quad \mu(\mathrm{h}\,\overline{\#}\,\varphi)(\mathrm{h}')=\sum\varphi(\mathrm{h}'_2\mathrm{p}^2_{\mathrm{L}})\mathrm{h}\mathrm{h}'_1\mathrm{p}^1_{\mathrm{L}}$$

for all $h,h'\in H$ and $\varphi\in H^*$, where $p_L=\sum p^1_L\otimes p^2_L$ is the element defined by (1.20). Then μ is a bijection, and therefore there exists a unique H-module algebra structure on $\mathrm{End_k(H)}$ such that μ becomes an H-module algebra isomorphism. The multiplication, the unit and the H-module structure of $\mathrm{End_k(H)}$ are given by

$$(u\overline{\circ}v)(h)=\sum u(v(hx^3X^3_2)S^{-1}(S(x^1X^2)\alpha x^2X^3_1))S^{-1}(X^1) \tag{2.13}$$

$$1_{\mathrm{End_k(H)}}=S^{-1}(\beta)\rightharpoonup id_H\quad;\quad(h\cdot u)(h')=u(h'h_2)S^{-1}(h_1) \tag{2.14}$$

for all $u,v\in\mathrm{End_k(H)}$ and $h,h'\in H$.

Proof. Let $\{e_i\}_{i=\overline{1,n}}$ be a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*. We claim that the inverse of μ is $\mu^{-1}:\ \mathrm{End_k(H)} \to \mathrm{H} \,\overline{\#}\, \mathrm{H}^*$ given by

$$\mu^{-1}(u) = \sum u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i$$

for all $u \in \mathrm{End_k(H)}$, where $q_L = \sum q_L^1 \otimes q_L^2$ is the element defined by (1.20). Indeed, for any $h \in H$ and $\varphi \in H^*$ we have:

$$\begin{aligned}
(\mu^{-1}\circ\mu)(h\,\overline{\#}\,\varphi) &= \sum_{i=1}^{n} \mu(h\,\overline{\#}\,\varphi)(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i \\
&= \sum_{i=1}^{n} \varphi((q_L^2)_2(e_i)_{(2,2)}p_L^2)h(q_L^2)_1(e_i)_{(2,1)}p_L^1 S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i \\
(1.22) \quad &= \sum_{i=1}^{n} \varphi((q_L^2)_2 p_L^2 e_i)h(q_L^2)_1 p_L^1 S^{-1}(q_L^1)\,\overline{\#}\,e^i \\
(1.24) \quad &= \sum_{i=1}^{n} \varphi(e_i)h\,\overline{\#}\,e^i = h\,\overline{\#}\,\varphi
\end{aligned}$$

and, in a similar way, for $u \in \mathrm{End_k(H)}$ and $h \in H$ we have that

$$\begin{aligned}
(\mu\circ\mu^{-1})(u)(h) &= \sum_{i=1}^{n} \mu(u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i)(h) \\
&= \sum_{i=1}^{n} e^i(h_2 p_L^2)u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)h_1 p_L^1 \\
&= \sum u(q_L^2 h_{(2,2)}(p_L^2)_2)S^{-1}(q_L^1 h_{(2,1)}(p_L^2)_1)h_1 p_L^1 \\
(1.22) \quad &= \sum u(hq_L^2(p_L^2)_2)S^{-1}(q_L^1(p_L^2)_1)p_L^1 \\
(1.24) \quad &= u(h)1_H = u(h).
\end{aligned}$$

Using the bijection μ, we transport the H-module algebra structure from $H\,\overline{\#}\,H^*$ to $\mathrm{End_k(H)}$. First we compute the transported multiplication $\overline{\circ}$: for all $u, v \in \mathrm{End_k(H)}$, we find

$$\begin{aligned}
u\overline{\circ}v &= \sum \mu(\mu^{-1}(u)\mu^{-1}(v)) \\
&= \sum_{i,j=1}^{n} \mu((u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i)(v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)\,\overline{\#}\,e^j)) \\
(2.11) \quad &= \sum_{i,j=1}^{n} \mu(u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)[v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)]_1 x^1 \\
&\quad \overline{\#}\,(e^i \leftharpoonup [v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)]_2 x^2)(e^j \leftharpoonup x^3))
\end{aligned}$$

where $\sum Q_L^1 \otimes Q_L^2$ is another copy of q_L. Note that (1.3) and (1.20) imply that

$$\sum S(x^1)q_L^1 x_1^2 \otimes q_L^2 x_2^2 \otimes x^3 = \sum q_L^1 X^1 \otimes (q_L^2)_1 X^2 \otimes (q_L^2)_2 X^3. \tag{2.15}$$

Hence, for all $h \in H$ we have that

$$\begin{aligned}
(u\overline{\circ}v)(h) \quad &\overset{(1.1)}{=} \sum_{i,j,l=1}^{n} < e^l, v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)h_1 >< e^i, (e_l)_2 x^2 (p_L^2)_1 > \\
&\quad < e^j, h_2 x^3 (p_L^2)_2 > u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)(e_l)_1 x^1 p_L^1
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,l=1}^{n} < e^l, v(Q_L^2 h_{(2,2)} x_2^3 (p_L^2)_{(2,2)}) S^{-1}(Q_L^1 h_{(2,1)} x_1^3 (p_L^2)_{(2,1)}) h_1 > \\
& < e^i, (e_l)_2 x^2 (p_L^2)_1 > u(q_L^2 (e_i)_2) S^{-1}(q_L^1 (e_i)_1)(e_l)_1 x^1 p_L^1 \\
(1.22) \quad &= \sum_{l=1}^{n} < e^l, v(hQ_L^2 x_2^3 (p_L^2)_{(2,2)}) S^{-1}(Q_L^1 x_1^3 (p_L^2)_{(2,1)}) > u(q_L^2 (e_l)_{(2,2)} (x^2 (p_L^2)_1)_2) \\
& S^{-1}(q_L^1 (e_l)_{(2,1)} (x^2 (p_L^2)_1)_1)(e_l)_1 x^1 p_L^1 \\
(1.22),(2.15) \quad &= \sum_{l=1}^{n} < e^l, v(hQ_L^2 (q_L^2)_{(2,2)} X_2^3 ((p_L^2)_2)_2) S^{-1}(Q_L^1 (q_L^2)_{(2,1)} X_1^3 ((p_L^2)_2)_1) > \\
& u(e_l (q_L^2)_1 X^2 (p_L^2)_{(1,2)}) S^{-1}(q_L^1 X^1 (p_L^2)_{(1,1)}) p_L^1 \\
(1.1) \quad &= \sum u(v(hQ_L^2 [q_L^2 (p_L^2)_2]_{(2,2)} X_2^3) S^{-1}(Q_L^1 ([q_L^2 (p_L^2)_2]_{(2,1)} X_1^3)[q_L^2 (p_L^2)_2]_1 X^2) \\
& S^{-1}(q_L^1 (p_L^2)_1 X^1) p_L^1 \\
(1.24) \quad &= \sum u(v(hQ_L^2 X_2^3) S^{-1}(Q_L^1 X_1^3) X^2) S^{-1}(X^1) \\
&= \sum u(v(hx^3 X_2^3) S^{-1}(S(x^1 X^2) \alpha x^2 X_1^3)) S^{-1}(X^1).
\end{aligned}
$$

Thus, we have obtained (2.13). Similar computations show that the tranported unit and the H-action on $\mathrm{End_k(H)}$ are given by (2.14). □

Remarks 2.4 Let H be a finite dimensional quasi-Hopf algebra, $\{e_i\}_{i=\overline{1,n}}$ a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*.
1) The bijection μ defined in Proposition 2.3 induces an associative algebra structure on the k-vector space $H \otimes H^*$: it suffices to transport the composition on $\mathrm{End_k(H)}$ to $H \otimes H^*$. If H is a Hopf algebra, then we find the smash product, and the corresponding category of left or right representations is equivalent to a certain category of Hopf modules $\mathcal{M}_H^H$. This could open the door to defining the category of Hopf modules over a quasi-Hopf algebra. Unfortunately, the involved structures are not easy to describe, except in the situations where H is a twisted Hopf algebra. A possible way to define $\mathcal{M}_H^H$ will be presented in the next Section.
2) Let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H-comodule algebra. As in the Hopf case, it is possible to associate different (quasi)smash products to $\mathfrak{A}$. Observe first that the map $\nu : \mathfrak{A} \,\overline{\#}\, H^* \to \mathrm{Hom_k(H, \mathfrak{A})}$ given by $\nu(\mathfrak{a} \,\overline{\#}\, \varphi)(h) = \varphi(h)\mathfrak{a}$, for all $\mathfrak{a} \in \mathfrak{A}$, $\varphi \in H^*$ and $h \in H$, is a k-linear isomorphism. The inverse of ν is given by the formula

$$
\nu^{-1}(w) = \sum_{i=1}^{n} w(e_i) \,\overline{\#}\, e^i
$$

for $w \in \mathrm{Hom_k(H, \mathfrak{A})}$. Secondly, by transporting the quasi-smash algebra structure from $\mathfrak{A} \,\overline{\#}\, H^*$ to $\mathrm{Hom_k(H, \mathfrak{A})}$ via the isomorphism ν, we obtain that $\mathrm{Hom_k(H, \mathfrak{A})}$ is an H-module algebra. So, if H is an arbitrary quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra, then we can define the quasi-smash product $\overline{\#}\,(H, \mathfrak{A})$ as follows: $\overline{\#}\,(H, \mathfrak{A})$ is the k-vector space $Hom_k(H, \mathfrak{A})$ with multiplication given by

$$
(v * w)(h) = \sum v(w(\tilde{x}^3 h_2)_{<1>} \tilde{x}^2 h_1) w(\tilde{x}^3 h_2)_{<0>} \tilde{x}^1 \tag{2.16}
$$

for $v, w \in \overline{\#}\,(H, \mathfrak{A})$ and $h \in H$. The unit is $1_{\overline{\#}\,(H,\mathfrak{A})}(h) = \varepsilon(h) 1_{\mathfrak{A}}$ and the H-module structure is given by $(h \cdot v)(h') = v(h'h)$, $h, h' \in H$, $v \in \mathrm{Hom_k(H, \mathfrak{A})}$. Of course, if H is finite dimensional then $\mathfrak{A} \,\overline{\#}\, H^* \simeq \overline{\#}\,(H, \mathfrak{A})$ as H-module algebras.

3 Two-sided Hopf modules and relative Hopf modules

We cannot define comodules over a quasi-bialgebra H, because a quasi-bialgebra is not coassociative. Also the definition of Hopf modules is quite complicated. A possible approach is the following: assume that we can deform the comultiplication on H in such a way that H with this new comultiplication (denoted $\overline{H}$) becomes a coalgebra in the monoidal category $\mathcal{M}_H$, and define $\mathcal{M}_H^H$ as the category of right $[\overline{H},H]$-Hopf modules, as introduced in [3]. More precisely, a right H-Hopf module is a right H-module which is also a right $\overline{H}$-comodule in the monoidal category $\mathcal{M}_H$ (for more detail, see Section 5). This machinere works if H is a twisted bialgebra. Indeed, if H is an ordinary bialgebra, and $F \in H \otimes H$ is a twist, then H_F is a quasi-bialgebra with counit ε and comultiplication given by (1.8). If we define on H_F a new comultiplication given by

$$\overline{\Delta}_F(h) = \sum h_1 \mathcal{G}^1 \otimes h_2 \mathcal{G}^2$$

for all $h \in H$, where $F^{-1} = \sum \mathcal{G}^1 \otimes \mathcal{G}^2$, and we denote H_F with this new comultiplication by $\overline{H}_F$, then it is not hard to see that $\overline{H}_F$ is a coalgebra in the monoidal category $\mathcal{M}_{H_F}$. Therefore, by a right H_F-Hopf module, we mean a k-vector space M, which is a right H-module (the right H-action of $h \in H$ on $m \in M$ is denoted by mh), together with a k-linear map $\rho_M : M \to M \otimes H$ such that the following relations hold, for all $m \in M$ and $h \in H$:

$$\begin{aligned}
&(\rho_M \otimes id_H)(\rho_M(m))(F \otimes 1_H) = (id_M \otimes \Delta)(\rho_M(m)F),\\
&(id_M \otimes \varepsilon)(\rho_M(m)) = m,\\
&\rho_M(mh)F = \rho_M(m)F\Delta(h).
\end{aligned}$$

We can conclude that categories of Hopf modules over quasi-Hopf algebras can be defined using (co)algebras and (co)modules in monoidal categories. This point of view was used in [3], [14] and [19] in order to define the categories of relative Hopf modules, quasi-Hopf bimodules and two-sided two-cosided Hopf modules. In the sequel, we will study all these categories in a more general context.

Definition 3.1 *Let H be a quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. A two-sided $(\mathfrak{A},H)$-Hopf module is an $(\mathfrak{A},H)$-bimodule M together with a k-linear map*

$$\rho_M : M \to M \otimes H, \ \ \rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$$

satisfying the following relations, for all $m \in M$, $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$. The actions of $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$ on $m \in M$ are denoted by $h \succ m$ amd $m \prec \mathfrak{a}$.

$$(id_M \otimes \varepsilon) \circ \rho_M = id_M, \tag{3.1}$$

$$\Phi \cdot (\rho_M \otimes id_H)(\rho_M(m)) = (id_M \otimes \Delta)(\rho_M(m)) \cdot \Phi_\rho, \tag{3.2}$$

$$\rho_M(h \succ m) = \sum h_1 \succ m_{(0)} \otimes h_2 m_{(1)}, \tag{3.3}$$

$$\rho_M(m \prec \mathfrak{a}) = \sum m_{(0)} \prec \mathfrak{a}_{<0>} \otimes m_{(1)}\mathfrak{a}_{<1>}. \tag{3.4}$$

The category of two-sided $(\mathfrak{A},H)$-Hopf modules and left H-linear, right $\mathfrak{A}$-linear and right H-colinear maps is denoted by ${}_H\mathcal{M}_{\mathfrak{A}}^H$.
Observe that the category of two-sided (H,H)-Hopf bimodules is nothing else then the category of right quasi-Hopf H-bimodules introduced in [14].
We will use the following notation, similar to the notation for the comultiplication on a quasi-bialgebra:

$$\begin{aligned}
(\rho_M \otimes id_H)(\rho_M(m)) &= \sum m_{(0,0)} \otimes m_{(0,1)} \otimes m_{(1)},\\
(id_M \otimes \Delta_H)(\rho_M(m)) &= \sum m_{(0)} \otimes m_{(1)_1} \otimes m_{(1)_2}.
\end{aligned}$$

Examples 3.2 Let H be a quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra.

1) $\mathcal{V}=\mathfrak{A}\otimes H\in {}_H\mathcal{M}_{\mathfrak{A}}^H$. The structure maps are

$$h\succ(\mathfrak{a}\otimes h')=\mathfrak{a}\otimes hh' \quad ; \quad (\mathfrak{a}\otimes h)\prec\mathfrak{a}'=\sum \mathfrak{a}\mathfrak{a}'_{<0>}\otimes h\mathfrak{a}'_{<1>}$$

and

$$\rho_{\mathcal{V}}(\mathfrak{a}\otimes h)=\sum \mathfrak{a}\tilde{X}^1\otimes h_1\tilde{X}^2\otimes h_2\tilde{X}^3$$

for all $h,h'\in H$ and $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$. We leave verification of the detail to the reader.

2) $\mathcal{U}=H\otimes\mathfrak{A}\in {}_H\mathcal{M}_{\mathfrak{A}}^H$. Now the structure maps are given by the following formulas, for all $h,h'\in H$ and $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$:

$$h\succ(h'\otimes\mathfrak{a})=hh'\otimes\mathfrak{a} \quad ; \quad (h\otimes\mathfrak{a})\prec\mathfrak{a}'=h\otimes\mathfrak{a}\mathfrak{a}'$$

and

$$\rho_{\mathcal{U}}(h\otimes\mathfrak{a})=\sum h_1S^{-1}(q_L^2\tilde{X}_2^3g^2)\otimes\tilde{X}^1\mathfrak{a}_{<0>}\otimes h_2S^{-1}(q_L^1\tilde{X}_1^3g^1)\tilde{X}^2\mathfrak{a}_{<1>}. \tag{3.5}$$

Here $q_L=\sum q_L^1\otimes q_L^2$ and $f^{-1}=\sum g^1\otimes g^2$ are the elements defined by the formulas (1.20) and (1.16).

Now consider $\theta:\ \mathcal{V}\to\mathcal{U}$ given by

$$\theta(\mathfrak{a}\otimes h)=\sum hS^{-1}(\mathfrak{a}_{<1>}\tilde{p}_\rho^2)\otimes\mathfrak{a}_{<0>}\tilde{p}_\rho^1$$

for all $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$, where we use the notation

$$\tilde{p}_\rho=\sum\tilde{p}_\rho^1\otimes\tilde{p}_\rho^2=\sum\tilde{x}^1\otimes\tilde{x}^2\beta S(\tilde{x}^3)\in\mathfrak{A}\otimes H. \tag{3.6}$$

θ is bijective; its inverse $\theta^{-1}:\ \mathcal{U}\to\mathcal{V}$ is defined as follows

$$\theta^{-1}(h\otimes\mathfrak{a})=\sum\tilde{q}_\rho^1\mathfrak{a}_{<0>}\otimes h\tilde{q}_\rho^2\mathfrak{a}_{<1>}$$

with the notation

$$\tilde{q}_\rho=\sum\tilde{q}_\rho^1\otimes\tilde{q}_\rho^2=\sum\tilde{X}^1\otimes S^{-1}(\alpha\tilde{X}^3)\tilde{X}^2\in\mathfrak{A}\otimes H. \tag{3.7}$$

Furthermore, θ is a morphism of two-sided $(\mathfrak{A},H)$-Hopf bimodules, and we conclude that $\mathcal{U}=H\otimes\mathfrak{A}$ and $\mathfrak{A}\otimes H=\mathcal{V}$ are isomorphic in ${}_H\mathcal{M}_{\mathfrak{A}}^H$.
To prove this, we proceed as follows. First, by [12, Lemma 9.1], we have the following relations, for all $\mathfrak{a}\in\mathfrak{A}$:

$$\sum\rho(\mathfrak{a}_{<0>})\tilde{p}_\rho[1_{\mathfrak{A}}\otimes S(\mathfrak{a}_{<1>})]=\tilde{p}_\rho[\mathfrak{a}\otimes 1_H], \tag{3.8}$$
$$\sum[1_{\mathfrak{A}}\otimes S^{-1}(\mathfrak{a}_{<1>})]\tilde{q}_\rho\rho(\mathfrak{a}_{<0>})=[\mathfrak{a}\otimes 1_H]\tilde{q}_\rho, \tag{3.9}$$
$$\sum\rho(\tilde{q}_\rho^1)\tilde{p}_\rho[1_{\mathfrak{A}}\otimes S(\tilde{q}_\rho^2)]=1_{\mathfrak{A}}\otimes 1_H, \tag{3.10}$$
$$\sum[1_{\mathfrak{A}}\otimes S^{-1}(\tilde{p}_\rho^2)]\tilde{q}_\rho\rho(\tilde{p}_\rho^1)=1_{\mathfrak{A}}\otimes 1_H, \tag{3.11}$$
$$\Phi_\rho(\rho\otimes id_H)(\tilde{p}_\rho)\tilde{p}_\rho=\sum(id\otimes\Delta)(\rho(\tilde{x}^1)\tilde{p}_\rho)(1_{\mathfrak{A}}\otimes g^1S(\tilde{x}^3)\otimes g^2S(\tilde{x}^2)), \tag{3.12}$$
$$(\tilde{q}_\rho\otimes 1_H)(\rho\otimes id_H)(\tilde{q}_\rho)\Phi_\rho^{-1}=\sum[1_{\mathfrak{A}}\otimes S^{-1}(f^2\tilde{X}^3)\otimes S^{-1}(f^1\tilde{X}^2)](id_{\mathfrak{A}}\otimes\Delta)(\tilde{q}_\rho\rho(\tilde{X}^1)). \tag{3.13}$$

Here $f=\sum f^1\otimes f^2$ is the element defined in (1.15) and $f^{-1}=\sum g^1\otimes g^2$. Using (3.8-3.11) we can show easily that θ and θ^{-1} are inverses, and that $\mathcal{U}$ is an $(H,\mathfrak{A})$-bimodule via the actions $\succ$ and $\prec$. We will

finally compute the right H-coaction on $\mathcal{U}$ transported from the coaction on $\mathcal{V}$ using θ, and then see that it coincides with (3.5). First observe that (3.6,2.2) and (2.4) imply

$$\sum \tilde{X}^1_{<1>}\tilde{p}^2_\rho S(\tilde{X}^2)\otimes \tilde{X}^1_{<0>}\tilde{p}^1_\rho\otimes \tilde{X}^3 = \sum \tilde{x}^2 S(\tilde{x}^3_1 p^1_L)\otimes \tilde{x}^1\otimes \tilde{x}^3_2 p^2_L, \tag{3.14}$$

where $p_L = \sum p^1_L\otimes p^2_L$ is the element defined in (1.20). Therefore, for all $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$ we have:

$$\begin{aligned}
&(\theta\otimes id_H)\circ\rho_{\mathcal{V}}\circ\theta^{-1}(h\otimes\mathfrak{a})\\
&\quad= \sum(\theta\otimes id_H)(\tilde{q}^1_\rho\mathfrak{a}_{<0>}\tilde{X}^1\otimes h_1(\tilde{q}^2_\rho)_1\mathfrak{a}_{<1>_1}\tilde{X}^2\otimes h_2(\tilde{q}^2_\rho)_2\mathfrak{a}_{<1>_2}\tilde{X}^3)\\
&\quad= \sum h_1(\tilde{q}^2_\rho)_1\mathfrak{a}_{<1>_1}\tilde{X}^2 S^{-1}((\tilde{q}^1_\rho)_{<1>}\mathfrak{a}_{<0,1>}\tilde{X}^1_{<1>}\tilde{p}^2_\rho)\otimes(\tilde{q}^1_\rho)_{<0>}\mathfrak{a}_{<0,0>}\tilde{X}^1_{<0>}\tilde{p}^1_\rho\\
&\qquad\otimes h_2(\tilde{q}^2_\rho)_2\mathfrak{a}_{<1>_2}\tilde{X}^3\\
(3.14)\quad&= \sum h_1(\tilde{q}^2_\rho)_1\mathfrak{a}_{<1>_1}\tilde{x}^3_1 p^1_L S^{-1}((\tilde{q}^1_\rho)_{<1>}\mathfrak{a}_{<0,1>}\tilde{x}^2)\otimes(\tilde{q}^1_\rho)_{<0>}\mathfrak{a}_{<0,0>}\tilde{x}^1\\
&\qquad\otimes h_2(\tilde{q}^2_\rho)_2\mathfrak{a}_{<1>_2}\tilde{x}^3_2 p^2_L\\
(2.1)\quad&= \sum h_1(\tilde{q}^2_\rho)_1\tilde{x}^3_1\mathfrak{a}_{<1>_{(2,1)}}p^1_L S^{-1}((\tilde{q}^1_\rho)_{<1>}\tilde{x}^2\mathfrak{a}_{<1>_1})\otimes(\tilde{q}^1_\rho)_{<0>}\tilde{x}^1\mathfrak{a}_{<0>}\\
&\qquad\otimes h_2(\tilde{q}^2_\rho)_2\tilde{x}^3_2\mathfrak{a}_{<1>_{(2,2)}}p^2_L\\
(1.22)\quad&= \sum h_1(\tilde{q}^2_\rho)_1\tilde{x}^3_1 p^1_L S^{-1}((\tilde{q}^1_\rho)_{<1>}\tilde{x}^2)\otimes(\tilde{q}^1_\rho)_{<0>}\tilde{x}^1\mathfrak{a}_{<0>}\otimes h_2(\tilde{q}^2_\rho)_2\tilde{x}^3_2 p^2_L\mathfrak{a}_{<1>}\\
(3.13)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho(\tilde{Q}^1_\rho)_{<1>}\tilde{z}^2(\tilde{y}^1_{<1>}\tilde{x}^3)_1 p^1_L S^{-1}((\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>}\tilde{z}^1\tilde{y}^1_{<0>})_{<1>}\tilde{x}^2)\\
&\qquad\otimes(\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>}\tilde{z}^1\tilde{y}^1_{<0>})_{<0>}\tilde{x}^1\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{z}^3(\tilde{y}^1_{<1>}\tilde{x}^3)_2 p^2_L\mathfrak{a}_{<1>}\\
(2.1)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho(\tilde{Q}^1_\rho)_{<1>}\tilde{z}^2\tilde{x}^3_1\tilde{y}^1_{<1>_{(2,1)}}p^1_L S^{-1}((\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<1>}\tilde{z}^1_{<1>}\tilde{x}^2\tilde{y}^1_{<1>_1})\\
&\qquad\otimes(\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<0>}\tilde{z}^1_{<0>}\tilde{x}^1\tilde{y}^1_{<0>}\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{z}^3\tilde{x}^3_2\tilde{y}^1_{<1>_{(2,2)}}p^2_L\mathfrak{a}_{<1>}\\
(1.22)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho(\tilde{Q}^1_\rho)_{<1>}\tilde{z}^2\tilde{x}^3_1 p^1_L S^{-1}((\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<1>}\tilde{z}^1_{<1>}\tilde{x}^2)\\
&\qquad\otimes(\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<0>}\tilde{z}^1_{<0>}\tilde{x}^1\tilde{y}^1_{<0>}\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{z}^3\tilde{x}^3_2 p^2_L\tilde{y}^1_{<1>}\mathfrak{a}_{<1>}\\
(1.20,2.2)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho(\tilde{Q}^1_\rho)_{<1>}\tilde{x}^3\tilde{z}^2_2 S^{-1}((\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<1>}\tilde{x}^2\tilde{z}^2_1)\\
&\qquad\otimes(\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>})_{<0>}\tilde{x}^1\tilde{z}^1\tilde{y}^1_{<0>}\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{z}^3\tilde{y}^1_{<1>}\mathfrak{a}_{<1>}\\
(2.4,2.1)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho\tilde{x}^3(\tilde{Q}^1_\rho)_{<1>_2}S^{-1}((\tilde{q}^1_\rho)_{<1>}\tilde{x}^2(\tilde{Q}^1_\rho)_{<1>_1}\beta)\\
&\qquad\otimes(\tilde{q}^1_\rho)_{<0>}\tilde{x}^1(\tilde{Q}^1_\rho)_{<0>}\tilde{y}^1_{<0>}\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{y}^1_{<1>}\mathfrak{a}_{<1>}\\
(2.3,3.6)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 g^2)\tilde{q}^2_\rho S^{-1}((\tilde{q}^1_\rho)_{<1>}\tilde{p}^2_\rho)\otimes(\tilde{q}^1_\rho)_{<0>}\tilde{p}^1_\rho\tilde{Q}^1_\rho\tilde{y}^1_{<0>}\mathfrak{a}_{<0>}\\
&\qquad\otimes h_2 S^{-1}(\tilde{y}^2 g^1)\tilde{Q}^2_\rho\tilde{y}^1_{<1>}\mathfrak{a}_{<1>}\\
(3.10,2.2)\quad&= \sum h_1 S^{-1}(\tilde{y}^3 x^3\tilde{X}^3_2 g^2)\otimes\tilde{y}^1\tilde{X}^1\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(\alpha\tilde{y}^2_2 x^2\tilde{X}^3_1 g^1)\tilde{y}^2_1 x^1\tilde{X}^2\mathfrak{a}_{<1>}\\
(2.3,1.20)\quad&= \sum h_1 S^{-1}(q^2_L\tilde{X}^3_2 g^2)\otimes\tilde{X}^1\mathfrak{a}_{<0>}\otimes h_2 S^{-1}(q^1_L\tilde{X}^3_1 g^1)\tilde{X}^2\mathfrak{a}_{<1>}\\
&= \rho_{\mathcal{U}}(h\otimes\mathfrak{a})
\end{aligned}$$

as needed.

The aim of this Section is to prove that, over a finite dimensional quasi-Hopf algebra H, the category ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to a certain category of relative Hopf modules defined in [3]. Let H be a finite dimensional quasi-bialgebra and A a left H-module algebra. Recall that a k-vector space M is called a right (H^*,A)-Hopf module if M is a right H^*-comodule and a right A-module in the monoidal category of right H^*-comodules $\mathcal{M}^{H^*}$. In terms of H this means:

- M is a left H-module; denote the action of $h\in H$ on $m\in M$ by $h\bullet m$;

- A acts on M from the right; denote the action of $a \in A$ on $m \in M$ by $m \bullet a$;

- for all $m \in M$, $h \in H$ and $a, a' \in A$, we have

$$m \bullet 1_A = m,$$
$$(m \bullet a) \bullet a' = \sum (X^1 \bullet m) \bullet [(X^2 \cdot a)(X^3 \cdot a')], \tag{3.15}$$
$$h \bullet (m \bullet a) = \sum (h_1 \bullet m) \bullet (h_2 \cdot a). \tag{3.16}$$

$\mathcal{M}_A^{H^*}$ will be the category of right (H^*, A)-Hopf modules and A-linear H^*-colinear maps.
Before we can establish the claimed isomorphism of categories, we need a few Lemmas.

Lemma 3.3 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. We have a functor*

$$F : {}_H\mathcal{M}_{\mathfrak{A}}^H \to \mathcal{M}_{\mathfrak{A} \,\overline{\#}\, H^*}^{H^*}.$$

For $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$, $F(M) = M$, with structure maps

- *M is a left H-module via $h \bullet m = S^2(h) \succ m$, $m \in M$, $h \in H$;*

- *$\mathfrak{A} \,\overline{\#}\, H^*$ acts on M from the right by*

$$m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi) = \sum \varphi S^{-1}(S(U^1) f^2 m_{(1)} \mathfrak{a}_{<1>} \tilde{p}_\rho^2) S(U^2) f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>} \tilde{p}_\rho^1 \tag{3.17}$$

where we denote

$$U = \sum U^1 \otimes U^2 = \sum g^1 S(q_R^2) \otimes g^2 S(q_R^1). \tag{3.18}$$

Proof. The most difficult par of the proof is to show that $F(M)$ satisfies the relations (3.15) and (3.16). It is then straightforward to show that a map in ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is also a map in $\mathcal{M}_{\mathfrak{A} \,\overline{\#}\, H^*}^{H^*}$, and that F is a functor.
By [14, Lemma 3.13] we have, for all $h \in H$:

$$U[1 \otimes S(h)] = \sum \Delta(S(h_1)) U (h_2 \otimes 1), \tag{3.19}$$
$$\Phi^{-1}(id \otimes \Delta)(U)(1 \otimes U) = (\Delta \otimes id)(\Delta(S(X^1))U)(X^2 \otimes X^3 \otimes 1). \tag{3.20}$$

Write $f = \sum f^1 \otimes f^2 = \sum F^1 \otimes F^2$, $f^{-1} = \sum g^1 \otimes g^2$, $\tilde{p}_\rho = \sum \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \sum \tilde{P}_\rho^1 \otimes \tilde{P}_\rho^1$, and $U = \sum U^1 \otimes U^2 = \sum \mathbf{U}^1 \otimes \mathbf{U}^2$. For all $m \in M$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, and $\varphi, \psi \in H^*$, we compute that

$$\begin{aligned}
&(X^1 \bullet m) \bullet \{[X^2 \cdot (\mathfrak{a} \,\overline{\#}\, \varphi)][X^3 \cdot (\mathfrak{a}' \,\overline{\#}\, \psi)]\} \\
&\quad = \sum (S^2(X^1) \succ m) \bullet [(\mathfrak{a} \,\overline{\#}\, X^2 \rightharpoonup \varphi)(\mathfrak{a}' \,\overline{\#}\, X^3 \rightharpoonup \psi)] \\
&\quad = \sum (S^2(X^1) \succ m) \bullet [\mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1 \,\overline{\#}\, (X^2 \rightharpoonup \varphi \leftharpoonup \mathfrak{a}'_{<1>} \tilde{x}^2)(X^3 \rightharpoonup \psi \leftharpoonup \tilde{x}^3)] \\
&\quad = \sum < (X^2 \rightharpoonup \varphi \leftharpoonup \mathfrak{a}'_{<1>} \tilde{x}^2)(X^3 \rightharpoonup \psi \leftharpoonup \tilde{x}^3), S^{-1}(S(U^1) f^2 S^2(X^1)_2 m_{(1)} \\
&\qquad (\mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1)_{<1>} \tilde{p}_\rho^2) > S(U^2) f^1 S^2(X^1)_1 \succ m_{(0)} \prec (\mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1)_{<0>} \tilde{p}_\rho^1 \\
(1.11) \quad &= \sum < \varphi, S^{-1}(F^2 S(U^1)_2 S(S(X^1)_1)_2 f_2^2 m_{(1)_2} \mathfrak{a}_{<1>_2} \mathfrak{a}'_{<0,1>_2} \tilde{x}^1_{<1>_2} (\tilde{p}_\rho^2)_2 \\
&\qquad g^2 S(\mathfrak{a}'_{<1>} \tilde{x}^2)) X^2 > < \psi, S^{-1}(F^1 S(U^1)_1 S(S(X^1)_1)_1 f_1^2 m_{(1)_1} \mathfrak{a}_{<1>_1} \\
&\qquad \mathfrak{a}'_{<0,1>_1} \tilde{x}^1_{<1>_1} (\tilde{p}_\rho^2)_1 g^1 S(\tilde{x}^3)) X^3 > \\
&\qquad S(S(X^1)_2 U^2) f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>} \mathfrak{a}'_{<0,0>} \tilde{x}^1_{<0>} \tilde{p}_\rho^1 \\
(1.11, 3.13, 2.1) \quad &= \sum \varphi S^{-1}(S(S(X^1)_{(1,1)} U_1^1 X^2) F^2 f_2^2 m_{(1)_2} \mathfrak{a}_{<1>_2} \tilde{X}^3 \mathfrak{a}'_{<0,1>} \tilde{p}_\rho^2 S(\mathfrak{a}'_{<1>}))
\end{aligned}$$

$$
\begin{aligned}
& \psi S^{-1}(S(S(X^1)_{(1,2)}U^1_2X^3)F^1 f^2_1 m_{(1)_1}\mathfrak{a}_{<1>_1}\tilde{X}^2(\mathfrak{a}'_{<0,0>}\tilde{p}^1_\rho)_{<1>}\tilde{P}^2_\rho) \\
& S(S(X^1)_2U^2)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{X}^1(\mathfrak{a}'_{<0,0>}\tilde{p}^1_\rho)_{<0>}\tilde{P}^1_\rho \\
(3.20,3.8) \quad = \quad & \sum \varphi S^{-1}(S(x^1U^1)F^2 f^2_2 m_{(1)_2}\mathfrak{a}_{<1>_2}\tilde{X}^3\tilde{p}^2_\rho) \\
& \psi S^{-1}(S(x^2U^2_1\mathbf{U}^1)F^1 f^2_1 m_{(1)_1}\mathfrak{a}_{<1>_1}\tilde{X}^2(\tilde{p}^1_\rho\mathfrak{a}')_{<1>}\tilde{P}^2_\rho) \\
& S(x^3U^2_2\mathbf{U}^2)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{X}^1(\tilde{p}^1_\rho\mathfrak{a}')_{<0>}\tilde{P}^2_\rho \\
(1.9,1.18,2.1) \quad = \quad & \sum \varphi S^{-1}(S(U^1)F^2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)\psi S^{-1}(S(U^2_1\mathbf{U}^1)f^2F^1_2 m_{(0,1)}\mathfrak{a}_{<0,1>} \\
& (\tilde{p}^1_\rho\mathfrak{a}')_{<1>}\tilde{P}^2_\rho)S(U^2_2\mathbf{U}^2)f^1F^1_1 \succ m_{(0,0)} \prec \mathfrak{a}_{<0,0>}(\tilde{p}^1_\rho\mathfrak{a}')_{<0>}\tilde{P}^1_\rho \\
(1.11,3.17) \quad = \quad & \sum \varphi S^{-1}(S(U^1)F^2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)(S(U^2)F^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{p}^1_\rho) \bullet (\mathfrak{a}' \,\overline{\#}\, \psi) \\
(3.17) \quad = \quad & [m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi)] \bullet (\mathfrak{a}' \,\overline{\#}\, \psi).
\end{aligned}
$$

Similar computations show that

$$
\begin{aligned}
& \sum (h_1 \bullet m) \bullet (h_2 \cdot (\mathfrak{a} \,\overline{\#}\, \varphi)) \\
& \quad = \quad \sum (S^2(h_1) \succ m) \bullet (\mathfrak{a} \,\overline{\#}\, h_2 \rightharpoonup \varphi) \\
& \quad = \quad \sum < \varphi, S^{-1}(S(U^1)f^2S^2(h_1)_2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)h_2 > \\
& \qquad S(U^2)f^1S^2(h_1)_1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{p}^1_\rho \\
(1.11,3.19) \quad = \quad & \sum \varphi S^{-1}(S(U^1)f^2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)S^2(h)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{p}^1_\rho \\
(3.17) \quad = \quad & h \bullet [m \bullet (\mathfrak{a} \,\overline{\#}\, \varphi)].
\end{aligned}
$$

□

Let us next discuss the construction in the converse direction.

Lemma 3.4 *Let H be a finite dimensional quasi-Hopf algebra, $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra and M a right $(\mathfrak{A} \,\overline{\#}\, H^*, H^*)$-Hopf module. Then we have a functor*

$$
G: \ \mathcal{M}^{H^*}_{\mathfrak{A} \,\overline{\#}\, H^*} \to {}_H\mathcal{M}^H_{\mathfrak{A}}.
$$

For $M \in \mathcal{M}^{H^*}_{\mathfrak{A} \,\overline{\#}\, H^*}$, $G(M) = M$, with structure maps ($h \in H$, $m \in M$, $\mathfrak{a} \in \mathfrak{A}$):

- $h \succ m = S^{-2}(h) \bullet m$;
- $m \prec \mathfrak{a} = m \bullet (\mathfrak{a} \,\overline{\#}\, \varepsilon)$;
- $\rho_M : M \to M \otimes H$ *is given by*

$$
\rho_M(m) = \sum m_{\{0\}} \otimes m_{\{1\}} = \sum_{i=1}^{n} [S^{-1}(V^2g^2) \bullet m] \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho) \otimes e_i \tag{3.21}
$$

where $\{e_i\}_{i=\overline{1,n}}$ and $\{e^i\}_{i=\overline{1,n}}$ are dual bases and

$$
V = \sum V^1 \otimes V^2 = \sum S^{-1}(f^2p^2_R) \otimes S^{-1}(f^1p^1_R). \tag{3.22}
$$

Proof. As in the previous part, the main thing to show is that $G(M)$ is an object of ${}_H\mathcal{M}^H_{\mathfrak{A}}$. It is then straightforward to show that G behaves well on the level of the morphisms (G is the identity on the morphisms).
From the fact that S^{-2} is an algebra map, it follows that M is a left H-module via the action $h \succ m = S^{-2}(h) \bullet m$. Take the map

$$i:\ \mathfrak{A} \to \mathfrak{A} \,\overline{\#}\, H^*, \quad i(\mathfrak{a}) = \mathfrak{a} \,\overline{\#}\, \varepsilon$$

for all $\mathfrak{a} \in \mathfrak{A}$. Then i is injective map, $i(1_{\mathfrak{A}}) = 1_{\mathfrak{A} \,\overline{\#}\, H^*}$, and $i(\mathfrak{a}\mathfrak{a}') = i(\mathfrak{a})i(\mathfrak{a}')$, for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$. Therefore, M becomes a right $\mathfrak{A}$-module by setting $m \prec \mathfrak{a} = m \bullet i(\mathfrak{a}) = m \bullet (\mathfrak{a} \,\overline{\#}\, \varepsilon)$, $m \in M$, $\mathfrak{a} \in \mathfrak{A}$. Moreover, it is not hard to see that, with this structure, M is an $(H, \mathfrak{A})$-bimodule. In order to check the relations (3.1-3.3) we need some formulas due to Hausser and Nill [12, Lemma 3.13], namely

$$[1 \otimes S^{-1}(h)]V = \sum (h_2 \otimes 1) V \Delta(S^{-1}(h_1)), \tag{3.23}$$

$$(\Delta \otimes id)(V)\Phi^{-1} = \sum (X^2 \otimes X^3 \otimes 1)(1 \otimes V)(id \otimes \Delta)(V\Delta(S^{-1}(X^1))). \tag{3.24}$$

Also, it is clear that

$$(\varphi \leftharpoonup h)S = S^{-1}(h) \rightharpoonup \varphi S \quad ; \quad (h \rightharpoonup \varphi)S = \varphi S \leftharpoonup S^{-1}(h) \tag{3.25}$$

for all $h \in H$ and $\varphi \in H^*$. Using (1.11), it follows that

$$(\varphi S)(\psi S) = \sum [(g^1 \rightharpoonup \psi \leftharpoonup f^1)(g^2 \rightharpoonup \varphi \leftharpoonup f^2)]S \tag{3.26}$$

for all $\varphi, \psi \in H^*$. Now, for any $h \in H$ and $m \in M$ we compute that

$$\begin{aligned}
&\sum h_1 \succ m_{\{0\}} \otimes h_2 m_{\{1\}} \\
&\quad = \sum_{i=1}^{n} S^{-2}(h_1) \bullet [(S^{-1}(V^2 g^2) \bullet m) \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho)] \otimes h_2 e_i \\
(3.16) \quad &= \sum_{i=1}^{n} [S^{-2}(h_1)_1 S^{-1}(V^2 g^2) \bullet m] \\
&\qquad \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-2}(h_1)_2 S^{-1}(V^1 g^1) \rightharpoonup (e^i \leftharpoonup h_2) S \leftharpoonup \tilde{q}^2_\rho) \otimes e_i \\
(1.11, 3.25) \quad &= \sum_{i=1}^{n} [S^{-1}(V^2 S^{-1}(h_1)_2 g^2) \bullet m] \\
&\qquad \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(h_2 V^1 S^{-1}(h_1)_1 g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho) \otimes e_i \\
(3.23) \quad &= \sum_{i=1}^{n} [S^{-1}(V^2 g^2) S^{-2}(h) \bullet m] \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho) \otimes e_i \\
&\quad = \rho_M(S^{-2}(h) \bullet m) = \rho_M(h \succ m),
\end{aligned}$$

and similarly, for any $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$

$$\begin{aligned}
&\sum m_{\{0\}} \prec \mathfrak{a}_{<0>} \otimes m_{\{1\}} \mathfrak{a}_{<1>} \\
&\quad = \sum_{i=1}^{n} [(S^{-1}(V^2 g^2) \bullet m) \bullet (\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho)] \bullet (\mathfrak{a}_{<0>} \,\overline{\#}\, \varepsilon) \otimes e_i \mathfrak{a}_{<1>} \\
(3.15, 2.11) \quad &= \sum_{i=1}^{n} (S^{-1}(V^2 g^2) \bullet m) \\
&\qquad \bullet (\tilde{q}^1_\rho \mathfrak{a}_{<0,0>} \,\overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup (\mathfrak{a}_{<1>} \rightharpoonup e^i) S \leftharpoonup \tilde{q}^2_\rho \mathfrak{a}_{<0,1>}) \otimes e_i
\end{aligned}$$

$$
\begin{aligned}
(3.25,3.8) &= \sum_{i=1}^{n}[S^{-1}(V^2g^2)\bullet m]\bullet(\mathfrak{a}\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup \tilde{q}^2_\rho)\otimes e_i\\
(2.11,3.15) &= \sum_{i=1}^{n}[(S^{-1}(V^2g^2)\bullet m)\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon)]\bullet(\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup \tilde{q}^2_\rho)\otimes e_i\\
(3.16) &= \rho_M(m\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon))=\rho_M(m\prec\mathfrak{a})
\end{aligned}
$$

so the relations (3.3) hold. (3.1) is obviously satisfied, thus remain to check (3.2) for our structures. For this we denote $f=\sum f^1\otimes f^2$, $f^{-1}=\sum g^1\otimes g^2=\sum G^1\otimes G^2$, $\tilde{q}_\rho=\sum\tilde{q}^1_\rho\otimes\tilde{q}^2_\rho=\sum\tilde{Q}^1_\rho\otimes\tilde{Q}^2_\rho$ and $V=\sum V^1\otimes V^2=\sum \mathbf{V}^1\otimes\mathbf{V}^2$. For all $m\in M$ we compute that

$$
\begin{aligned}
& X^1\succ m_{\{0,0\}}\otimes X^2m_{\{0,1\}}\otimes X^3m_{\{1\}}\\
(3.21,3.16) &= \sum_{i,j=1}^{n}\{S^{-2}(X^1)_1S^{-1}(\mathbf{V}^2G^2)\\
&\quad\bullet[(S^{-1}(V^2g^2)\bullet m)\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)]\}\\
&\quad\bullet(\tilde{Q}^1_\rho\,\overline{\#}\,S^{-2}(X^1)_2S^{-1}(\mathbf{V}^1G^1)\rightharpoonup e^jS\leftharpoonup\tilde{Q}^2_\rho)\otimes X^2e_j\otimes X^3e_i\\
(1.11) &= \sum_{i,j=1}^{n}\{S^{-1}(\mathbf{V}^2S^{-1}(X^1)_2G^2)\bullet[(S^{-1}(V^2g^2)\bullet m)\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\\
&\quad\rightharpoonup(e^i\leftharpoonup X^3)S\leftharpoonup\tilde{q}^2_\rho)]\}\bullet(\tilde{Q}^1_\rho\,\overline{\#}\,S^{-1}(\mathbf{V}^1S^{-1}(X^1)_1G^1)\\
&\quad\rightharpoonup(e^j\leftharpoonup X^2)S\leftharpoonup\tilde{Q}^2_\rho)\otimes e_j\otimes e_i\\
(1.11,3.16,3.25) &= \sum_{i,j=1}^{n}\{[S^{-1}(V^2\mathbf{V}^2_2S^{-1}(X^1)_{(2,2)}G^2_2g^2)\bullet m]\\
&\quad\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(X^3V^1\mathbf{V}^2_1S^{-1}(X^1)_{(2,1)}G^2_1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\}\\
&\quad\bullet(\tilde{Q}^1_\rho\,\overline{\#}\,S^{-1}(X^2\mathbf{V}^1S^{-1}(X^1)_1G^1)\rightharpoonup e^jS\leftharpoonup\tilde{Q}^2_\rho)\otimes e_j\otimes e_i\\
(3.24,3.15) &= \sum_{i,j=1}^{n}(X^1S^{-1}(V^2x^3G^2_2g^2)\bullet m)\bullet[(\tilde{q}^1_\rho\,\overline{\#}\,X^2S^{-1}(V^1_2x^2G^2_1g^1)\rightharpoonup e^iS\\
&\quad\rightharpoonup\tilde{q}^2_\rho)(\tilde{Q}^1_\rho\rightharpoonup X^3S^{-1}(V^1_1x^1G^1)\rightharpoonup e^jS\leftharpoonup\tilde{Q}^2_\rho)]\otimes e_j\otimes e_i\\
(1.9,1.18) &= \sum_{i,j=1}^{n}(S^{-1}(V^2g^2)\bullet m)\bullet[(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1_2g^1_2G^2)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\\
&\quad(\tilde{Q}^1_\rho\,\overline{\#}\,S^{-1}(V^1_1g^1_1G^1)\rightharpoonup e^jS\leftharpoonup\tilde{Q}^2_\rho)]\otimes e_j\otimes e_i\\
(1.11,2.11) &= \sum_{i,j=1}^{n}(S^{-1}(V^2g^2)\bullet m)\bullet[\tilde{q}^1_\rho(\tilde{Q}^1_\rho)_{<0>}\tilde{x}^1\,\overline{\#}\,(S^{-1}(V^1g^1)_1S^{-1}(G^2)\rightharpoonup e^iS\\
&\quad\leftharpoonup\tilde{q}^2_\rho(\tilde{Q}^1_\rho)_{<1>}\tilde{x}^2)(S^{-1}(V^1g^1)_2S^{-1}(G^1)\rightharpoonup e^jS\leftharpoonup\tilde{Q}^2_\rho\tilde{x}^3)]\otimes e_j\otimes e_i\\
(3.13,2.10) &= \sum_{i,j=1}^{n}[S^{-1}(V^2g^2)\bullet m]\bullet\{\tilde{q}^1_\rho\tilde{X}^1_{<0>}\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup[(S^{-1}(G^2)\rightharpoonup e^iS\\
&\quad\leftharpoonup S^{-1}(f^2\tilde{X}^3))(S^{-1}(G^1)\rightharpoonup e^jS\leftharpoonup S^{-1}(f^1\tilde{X}^2))]\leftharpoonup\tilde{q}^2_\rho\tilde{X}^1_{<1>}\}\otimes e_j\otimes e_i\\
(2.11,3.25) &= \sum_{i,j=1}^{n}[S^{-1}(V^2g^2)\bullet m]\bullet\{[\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup(f^2\tilde{X}^3\rightharpoonup e^i\leftharpoonup G^2)S\\
&\quad(f^1\tilde{X}^2\rightharpoonup e^j\leftharpoonup G^1)S\leftharpoonup\tilde{q}^2_\rho](\tilde{X}^1\,\overline{\#}\,\varepsilon)\}\otimes e_j\otimes e_i
\end{aligned}
$$

$$
\begin{aligned}
(3.26,3.15) &= \sum_{i,j=1}^{n} [(S^{-1}(V^2g^2)\bullet m)\bullet(\tilde{q}^1_\rho \,\overline{\#}\, S^{-1}(V^1g^1) \rightharpoonup (e^je^i)S \leftharpoonup \tilde{q}^2_\rho)] \\
&\quad \bullet(\tilde{X}^1 \,\overline{\#}\, \varepsilon)\otimes e_j\tilde{X}^2\otimes e_i\tilde{X}^3 \\
(3.21) &= \sum m_{\{0\}} \prec \tilde{X}^1 \otimes m_{\{1\}_1}\tilde{X}^2 \otimes m_{\{1\}_2}\tilde{X}^3
\end{aligned}
$$

and this finishes the proof. □

We are now able to prove the main result of this Section, generalizing [7, Proposition 2.3].

Theorem 3.5 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then the category of two-sided $(H,\mathfrak{A})$-Hopf modules ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to the category of right $(H^*,\mathfrak{A}\,\overline{\#}\,H^*)$-Hopf modules $\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$.*

Proof. We have to show that the functors F and G from Lemmas 3.3 and 3.4 are inverses.
First, let $M\in {}_H\mathcal{M}^H_{\mathfrak{A}}$. The structures on $G(F(M))$ (using first Lemma 3.3 and then Lemma 3.4) are denoted by $\succ'$, $\prec'$ and ρ'_M. For any $m\in M$, $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$ we have that

$$
\begin{aligned}
h\succ' m &= S^{-2}(h)\bullet m = S^2(S^{-2}(h))\succ m = h\succ m \\
m\prec' \mathfrak{a} &= m\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon) = m\prec\mathfrak{a}
\end{aligned}
$$

because $\sum\varepsilon(U^1)U^2=\sum\varepsilon(f^2)f^1=1$ and $\sum\varepsilon(m_{(1)})m_{(0)}=m$, $\sum\varepsilon(\mathfrak{a}_{<1>})\mathfrak{a}_{<0>}=\mathfrak{a}$. In order to prove that $\rho'_M=\rho_M$, observe first that $\sum g^1S(g^2\alpha)=\beta$, where we write $f^{-1}=\sum g^1\otimes g^2$. This equality together with (3.18,1.9) and (1.18) implies

$$
\sum g^2_2U^2\otimes g^1S(g^2_1U^1)=\sum p^2_L\otimes S(p^1_L) \tag{3.27}
$$

where $p_L=\sum p^1_L\otimes p^2_L$ is the element defined by (1.20). Secondly, by $\sum S^{-1}(f^2)\beta f^1=S^{-1}(\alpha)$, (1.9) and (1.18) we have that

$$
\sum S(p^2_L)f^1F^1_1\otimes S^{-1}(F^2)S(p^1_L)f^2F^1_2=q_R \tag{3.28}
$$

were $\sum F^1\otimes F^2$ is another copy of f, and q_R is the element defined by (1.19). Finally, from (3.27,3.28) and (1.23), it follows that

$$
\sum S(g^2_2U^2)f^1F^1_1(p^1_R)_1\otimes S^{-1}(F^2p^2_R)g^1S(g^2_1U^1)f^2F^1_2(p^1_R)_2=1\otimes 1. \tag{3.29}
$$

We now compute for $m\in M$ that

$$
\begin{aligned}
\rho'_M(m) &= \sum_{i=1}^{n}[S^{-1}(V^2g^2)\bullet m]\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\otimes e_i \\
&= \sum_{i=1}^{n}[S(V^2g^2)\succ m]\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\otimes e_i \\
(3.17) &= \sum_{i=1}^{n} < S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho, S^{-1}(S(U^1)f^2S(V^2g^2)_2m_{(1)}(\tilde{q}^1_\rho)_{<1>}\tilde{p}^2_\rho) > \\
&\quad S(U^2)f^1S(V^2g^2)_1\succ m_{(0)}\prec(\tilde{q}^1_\rho)_{<0>}\tilde{p}^1_\rho\otimes e_i \\
(1.11) &= \sum S(V^2_2g^2_2U^2)f^1\succ m_{(0)}\prec(\tilde{q}^1_\rho)_{<0>}\tilde{p}^1_\rho\otimes V^1g^1S(V^2_1g^2_1U^1)f^2 \\
&\quad m_{(1)}(\tilde{q}^1_\rho)_{<1>}\tilde{p}^2_\rho S(\tilde{q}^2_\rho) \\
(3.10) &= \sum S(V^2_2g^2_2U^2)f^1\succ m_{(0)}\otimes V^1g^1S(V^2_1g^2_1U^1)f^2m_{(1)} \\
(3.22,1.11) &= \sum S(g^2_2U^2)f^1F^1_1(p^1_R)_1\succ m_{(0)}\otimes S^{-1}(F^2p^2_R)g^1S(g^2_1U^1)f^2F^1_2(p^1_R)_2m_{(1)} \\
(3.29) &= \sum m_{(0)}\otimes m_{(1)}=\rho_M(m).
\end{aligned}
$$

and this finishes the proof of the fact that $G(F(M)) = M$.

Conversely, take $M \in \mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$. We want to show that $F(G(M)) = M$. Denote the left H-action and the right $\mathfrak{A}$-action on $F(G(M))$ by $\bullet'$. Using Lemmas 3.3 and 3.4, we find, for all $h \in H$ and $m \in M$:

$$h \bullet' m = S^2(h) \succ m = S^{-2}(S^2(h)) \bullet m = h \bullet m.$$

The proof of the fact that the right $\mathfrak{A}$-action $\bullet$ and $\bullet'$ on M coincide is a somewhat more complicated. Since $\sum f^2 S^{-1}(f^1\beta) = S(\alpha)$, (1.9) and (1.18) imply

$$\sum F^1 f^1_1 p^1_R \otimes f^2 S^{-1}(F^2 f^1_2 p^2_R) = \sum S(q^2_L) \otimes q^1_L \tag{3.30}$$

where $q_L = \sum q^1_L \otimes q^2_L$ is the element defined by (1.20). Also, by (1.9), (1.18) and using $\sum S(g^1)\alpha g^2 = S(\beta)$ we can prove the following relation

$$\sum S(G^1) q^1_L G^2_1 g^1 \otimes q^2_L G^2_2 g^2 = \sum S(p^2_R) \otimes S(p^1_R) \tag{3.31}$$

where $\sum G^1 \otimes G^2$ is another copy of f^{-1}. Now, from (3.18,1.11,3.30,3.31) and (1.23) it follows that

$$\sum S^{-1}(F^1 f^1_1 p^1_R) U^2_2 g^2 \otimes S(U^1) f^2 S^{-1}(F^2 f^1_2 p^2_R) U^2_1 g^1 = 1 \otimes 1. \tag{3.32}$$

Therefore, for all $m \in M$, $\mathfrak{a} \in \mathfrak{A}$ and $\varphi \in H^*$ we have that

$$\begin{aligned}
& m \bullet' (\mathfrak{a}\,\overline{\#}\,\varphi) \\
(3.17) \quad &= \sum \varphi S^{-1}(S(U^1) f^2 m_{\{1\}} \mathfrak{a}_{<1>} \tilde{p}^2_\rho) S(U^2) f^1 \succ m_{\{0\}} \prec \mathfrak{a}_{<0>} \tilde{p}^1_\rho \\
(3.21, 3.15, 2.11) \quad &= \sum_{i=1}^{n} \varphi S^{-1}(S(U^1) f^2 e_i \mathfrak{a}_{<1>} \tilde{p}^2_\rho) S^{-2}(S(U^2) f^1) \bullet \{[S^{-1}(V^2 g^2) \bullet m] \\
& \quad \bullet [\tilde{q}^1_\rho \mathfrak{a}_{<0,0>} (\tilde{p}^1_\rho)_{<0>} \,\overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho \mathfrak{a}_{<0,1>} (\tilde{p}^1_\rho)_{<1>}]\} \\
&= \sum_{i=1}^{n} \varphi(e_i) S^{-2}(S(U^2) f^1) \bullet \{[S^{-1}(V^2 g^2) \bullet m] \bullet [\tilde{q}^1_\rho \mathfrak{a}_{<0,0>} (\tilde{p}^1_\rho)_{<0>} \\
& \quad \overline{\#}\, S^{-1}(V^1 g^1) \rightharpoonup (\mathfrak{a}_{<1>} \tilde{p}^2_\rho \rightharpoonup e^i S^{-1} \leftharpoonup S(U^1) f^2) S \\
& \quad \leftharpoonup \tilde{q}^2_\rho \mathfrak{a}_{<0,1>} (\tilde{p}^1_\rho)_{<1>}]\} \\
(3.25, 3.8, 3.10) \quad &= \sum S^{-2}(S(U^2) f^1) \bullet \{[S^{-1}(V^2 g^2) \bullet m] \\
& \quad \bullet [\mathfrak{a}\,\overline{\#}\, S^{-1}(S(U^1) f^2 V^1 g^1) \rightharpoonup \varphi]\} \\
(3.16, 1.11) \quad &= \sum [S^{-1}(V^2 S^{-1}(S(U^2) f^1)_2 g^2) \bullet m] \\
& \quad \bullet [\mathfrak{a}\,\overline{\#}\, S^{-1}(S(U^1) f^2 V^1 S^{-1}(S(U^2) f^1)_1 g^1)) \rightharpoonup \varphi] \\
(3.22, 1.11) \quad &= \sum [S^{-1}(S^{-1}(F^1 f^1_1 p^1_R) U^2_2 g^2) \bullet m] \\
& \quad \bullet [\mathfrak{a}\,\overline{\#}\, S^{-1}(S(U^1) f^2 S^{-1}(F^2 f^1_2 p^2_R) U^2_1 g^1) \rightharpoonup \varphi] \\
(3.32) \quad &= m \bullet (\mathfrak{a}\,\overline{\#}\,\varphi)
\end{aligned}$$

and this finishes our proof. □

If H is a finite dimensional quasi-Hopf algebra and A is a left H-module algebra then the category $\mathcal{M}^{H^*}_A$ is isomorphic to the category of right modules over the smash product $A\#H$ ([3, Proposition 2.7]). Let M be a right $A\#H$-module, and denote the right action of $a\#h \in A\#H$ on $m \in M$ by $m \leftarrow (a\#h)$. Following [3], M is a right (H^*, A)-Hopf module, with structure maps

$$h \bullet m = m \leftharpoonup (1\#S(h)), \;\; m \bullet a = \sum m \leftharpoonup [g^1 S(q_R^2) \cdot a \# g^2 S(q_R^1)] \tag{3.33}$$

for all $m \in M$, $a \in A$ and $h \in H$. Conversely, if M is a right (H^*, A)-Hopf module then M is a right $A\#H$-module, with $A\#H$-action

$$m \leftharpoonup (a\#h) = \sum S^{-1}(h) \bullet [(S^{-1}(q_L^2 g^2) \bullet m) \bullet (S^{-1}(q_L^1 g^1) \cdot a)]. \tag{3.34}$$

Here $q_R = \sum q_R^1 \otimes q_R^2$, $q_L = \sum q_L^1 \otimes q_L^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are the elements defined by (1.19), (1.20) and (1.16). Combining this with Theorem 3.5, we obtain the following result.

Corollary 3.6 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. Then the category ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of right $(\mathfrak{A} \,\overline{\#}\, H^*)\#H$-modules, $\mathcal{M}_{(\mathfrak{A} \,\overline{\#}\, H^*)\#H}$.*

For later use, we describe the isomorphism of Corollary 3.6 explicitely, leaving verification of the details to the reader.
First take $M \in \mathcal{M}_{(\mathfrak{A} \,\overline{\#}\, H^*)\#H}$. The following structure maps make $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$:

$$h \succ m = m \leftharpoonup ((1_{\mathfrak{A}} \,\overline{\#}\, \varepsilon)\#S^{-1}(h)) \tag{3.35}$$

$$m \prec \mathfrak{a} = m \leftharpoonup ((\mathfrak{a} \,\overline{\#}\, \varepsilon)\#1) \tag{3.36}$$

$$\rho_M(m) = \sum_{i=1}^{n} [(\tilde{q}_\rho^1 \,\overline{\#}\, S^{-1}(g^2) \rightharpoonup e^i S \leftharpoonup \tilde{q}_\rho^2)\#S^{-1}(g^1)] \otimes e_i \tag{3.37}$$

for all $m \in M$, $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$. $\tilde{q}_\rho = \sum \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2$ is the element defined in (3.7), $\{e_i\}$ is a basis of H and $\{e^i\}$ is the corresponding dual basis of H^*.
Now take $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. Then M is a right $(\mathfrak{A} \,\overline{\#}\, H^*)\#H$-module via the action

$$m \leftharpoonup [(\mathfrak{a} \,\overline{\#}\, \varphi)\#h] = \sum \varphi S^{-1}(f^2 m_{(1)} \mathfrak{a}_{<1>} \tilde{p}_\rho^2) S(h) f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>} \tilde{p}_\rho^1. \tag{3.38}$$

In [14], it is shown that, for a finite dimensional quasi-Hopf algebra H, the category of right quasi-Hopf H-bimodules ${}_H\mathcal{M}_H^H$ naturally coincides with the category of representations of the two-sided crossed product $H \rtimes H^* \ltimes H$ constructed in [12]. We will show in Section 4 that $H \rtimes H^* \ltimes H = (H \,\overline{\#}\, H^*)\#H$ as algebras.

4 Two-sided crossed products are generalized smash products

Let H be a finite dimensional quasi-bialgebra, and $(\mathfrak{A}, \rho, \Phi_\rho)$,$(\mathfrak{B}, \lambda, \Phi_\lambda)$ respectively a right and a left H-comodule algebra. As in the case of a Hopf algebra, the right H-coaction (ρ, Φ_ρ) on $\mathfrak{A}$ induces a left H^*-action $\triangleright$: $H^* \otimes \mathfrak{A} \to \mathfrak{A}$ given by

$$\varphi \triangleright \mathfrak{a} = \sum \varphi(\mathfrak{a}_{<1>}) \mathfrak{a}_{<0>} \tag{4.1}$$

for all $\varphi \in H^*$ and $\mathfrak{a} \in \mathfrak{A}$, and where $\rho(\mathfrak{a}) = \sum a_{<0>} \otimes \mathfrak{a}_{<1>}$ for any $\mathfrak{a} \in \mathfrak{A}$. Similarly, the left H-action (λ, Φ_λ) on $\mathfrak{B}$ provides a right H^*-action $\triangleleft$: $\mathfrak{B} \otimes H^* \to \mathfrak{B}$ given by

$$\mathfrak{b} \triangleleft \varphi = \sum \varphi(\mathfrak{b}_{[-1]}) \mathfrak{b}_{[0]} \tag{4.2}$$

for all $\varphi \in H^*$ and $\mathfrak{b} \in \mathfrak{B}$, where we now denote $\lambda(\mathfrak{b}) = \sum \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0]}$ for $\mathfrak{b} \in \mathfrak{B}$. Following [12, Proposition 11.4 (ii)] we can define an algebra structure on the k-vector space $\mathfrak{A} \otimes H^* \otimes \mathfrak{B}$. This algebra is denoted by $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$ and its multiplication is given by

$$\begin{aligned}
&(\mathfrak{a} \rtimes \varphi \ltimes \mathfrak{b})(\mathfrak{a}' \rtimes \psi \ltimes \mathfrak{b}') \\
&\quad = \sum \mathfrak{a}(\varphi_1 \triangleright \mathfrak{a}')\tilde{x}^1_\rho \rtimes (\tilde{x}^1_\lambda \rightharpoonup \varphi_2 \leftharpoonup \tilde{x}^2_\rho)(\tilde{x}^2_\lambda \rightharpoonup \psi_1 \leftharpoonup \tilde{x}^3_\rho) \ltimes \tilde{x}^3_\lambda(\mathfrak{b} \triangleleft \psi_2)\mathfrak{b}'
\end{aligned} \tag{4.3}$$

for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$, and $\varphi, \psi \in H^*$, where we write $\mathfrak{a} \rtimes \varphi \ltimes \mathfrak{b}$ for $\mathfrak{a} \otimes \varphi \otimes \mathfrak{b}$ when viewed as an element of $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$. The unit of the algebra $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$ is $1_{\mathfrak{A}} \rtimes \varepsilon \ltimes 1_{\mathfrak{B}}$. Hausser and Nill called this algebra the two-sided crossed product. In this Section we will prove that this two-sided crossed product algebra is a generalized smash product between the quasi-smash product $\mathfrak{A} \,\overline{\#}\, H^*$ and $\mathfrak{B}$.

Proposition 4.1 *Let H be a quasi-bialgebra, A a left H-module algebra and $\mathfrak{B}$ a left H-comodule algebra. Let $A \blacktriangleright\!\!< \mathfrak{B} = A \otimes \mathfrak{B}$ as a k-module, with newly defined multiplication*

$$(a \blacktriangleright\!\!< \mathfrak{b})(a' \blacktriangleright\!\!< \mathfrak{b}') = \sum (\tilde{x}^1 \cdot a)(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a') \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]} \mathfrak{b}' \tag{4.4}$$

for all $a, a' \in A$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. Then $A \blacktriangleright\!\!< \mathfrak{B}$ is an associative algebra with unit $1_A \blacktriangleright\!\!< 1_{\mathfrak{B}}$.

Proof. For all $a, a', a'' \in A$ and $\mathfrak{b}, \mathfrak{b}', \mathfrak{b}'' \in \mathfrak{B}$ we have:

$$\begin{aligned}
&[(a \blacktriangleright\!\!< b)(a' \blacktriangleright\!\!< \mathfrak{b}')](a'' \blacktriangleright\!\!< \mathfrak{b}'') \\
&\quad = \sum [(\tilde{x}^1 \cdot a)(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a') \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]} \mathfrak{b}'](a'' \blacktriangleright\!\!< \mathfrak{b}'') \\
&\quad = \sum [(\tilde{y}^1_1 \tilde{x}^1 \cdot a)(\tilde{y}^1_2 \tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a')](\tilde{y}^2 \tilde{x}^3_{[-1]} \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a'') \blacktriangleright\!\!< \tilde{y}^3 \tilde{x}^3_{[0]} \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
(1.27) &\quad = \sum (X^1 \tilde{y}^1_1 \tilde{x}^1 \cdot a)[(X^2 \tilde{y}^1_2 \tilde{x}^2 \mathfrak{b}_{[-1]} \cdot a')(X^3 \tilde{y}^2 \tilde{x}^3_{[-1]} \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a'')] \\
&\qquad \blacktriangleright\!\!< \tilde{y}^3 \tilde{x}^3_{[0]} \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
(2.6) &\quad = \sum (\tilde{x}^1 \cdot a)[(\tilde{x}^2_1 \tilde{y}^1 \mathfrak{b}_{[-1]} \cdot a')(\tilde{x}^2_2 \tilde{y}^2 \mathfrak{b}_{[0,-1]} \mathfrak{b}'_{[-1]} \cdot a'')] \blacktriangleright\!\!< \tilde{x}^3 \tilde{y}^3 \mathfrak{b}_{[0,0]} \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
(2.5, 1.28) &\quad = \sum (\tilde{x}^1 \cdot a)\{(\tilde{x}^2 \mathfrak{b}_{[-1]} \cdot [(\tilde{y}^1 \cdot a')(\tilde{y}^2 \mathfrak{b}'_{[-1]} \cdot a'')]\} \blacktriangleright\!\!< \tilde{x}^3 \mathfrak{b}_{[0]} \tilde{y}^3 \mathfrak{b}'_{[0]} \mathfrak{b}'' \\
&\quad = \sum (a \blacktriangleright\!\!< \mathfrak{b})[(\tilde{y}^1 \cdot a')(\tilde{y}^2 \mathfrak{b}'_{[-1]} \cdot a'') \blacktriangleright\!\!< \tilde{y}^3 \mathfrak{b}'_{[0]} \mathfrak{b}''] \\
&\quad = (a \blacktriangleright\!\!< \mathfrak{b})[(a' \blacktriangleright\!\!< \mathfrak{b}')(a'' \blacktriangleright\!\!< \mathfrak{b}'')].
\end{aligned}$$

It follows from (2.7), (2.8) and (1.29) that $1_A \blacktriangleright\!\!< 1_{\mathfrak{B}}$ is the unit for $A \blacktriangleright\!\!< \mathfrak{B}$. □

Remark 4.2 Let H be a quasi-bialgebra and A a left H-module algebra. Then H is a left H-comodule algebra so it make sense to consider $A \blacktriangleright\!\!< H$. It is not hard to see that in this case $A \blacktriangleright\!\!< H$ is just the smash product $A\#H$. For this reason we will call the algebra $A \blacktriangleright\!\!< \mathfrak{B}$ in Proposition 4.1 the generalized smash product of A and $\mathfrak{B}$. In fact, our terminology is in agreement with the terminology used over Hopf algebras, see [9] and [5].

Let H be a finite dimensional quasi-bialgebra, $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. Then the quasi-smash product $\mathfrak{A} \,\overline{\#}\, H^*$ is a left H-module algebra so it makes sense to consider the generalized smash product $(\mathfrak{A} \,\overline{\#}\, H^*) \blacktriangleright\!\!< \mathfrak{B}$. The main result of this Section is now the following:

Proposition 4.3 *With notation as above, the algebras $(\mathfrak{A} \,\overline{\#}\, H^*) \blacktriangleright\!\!< \mathfrak{B}$ and $\mathfrak{A} \rtimes_\rho H^* \ltimes_\lambda \mathfrak{B}$ coincide.*

Proof. Using (4.4), (2.12) and (2.11) we compute that the multiplication on $(\mathfrak{A}\overline{\#}H^*)\blacktriangleright\!\!<\mathfrak{B}$ is given by

$$\begin{aligned}
&[(\mathfrak{a}\,\overline{\#}\,\varphi)\blacktriangleright\!\!<\mathfrak{b}][(\mathfrak{a}'\,\overline{\#}\,\psi)\blacktriangleright\!\!<\mathfrak{b}'] \\
&\quad= \sum[\tilde{x}^1_\lambda\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)][\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)]\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \\
&\quad= \sum(\mathfrak{a}\,\overline{\#}\,\tilde{x}^1_\lambda\rightharpoonup\varphi)(\mathfrak{a}'\,\overline{\#}\,\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\rightharpoonup\psi)\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \\
&\quad= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1_\rho\,\overline{\#}\,(\tilde{x}^1_\lambda\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2_\rho)(\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\rightharpoonup\psi\leftharpoonup\tilde{x}^3_\rho)\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \\
(4.1,4.2)\quad &= \sum\mathfrak{a}(\varphi_1\triangleright\mathfrak{a}')\tilde{x}^1_\rho\,\overline{\#}\,(\tilde{x}^1_\lambda\rightharpoonup\varphi_2\leftharpoonup\tilde{x}^2_\rho)(\tilde{x}^2_\lambda\rightharpoonup\psi_1\leftharpoonup\tilde{x}^3_\rho)\blacktriangleright\!\!<\tilde{x}^3_\lambda(\mathfrak{b}'\triangleleft\psi_2)\mathfrak{b}'
\end{aligned}$$

for $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$, $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$, and $\varphi,\psi\in H^*$. This is just the multiplication rule on the two-sided crossed product $\mathfrak{A}\rtimes_\rho H^*\ltimes_\lambda\mathfrak{B}$. □

Take $\mathfrak{B}=H$ in Proposition 4.3. From Remark 4.2, we obtain:

Corollary 4.4 *Let H be a finite dimensional quasi-bialgebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then $(\mathfrak{A}\,\overline{\#}\,H^*)\#H=\mathfrak{A}\rtimes_\rho H^*\ltimes_\Delta H$ as algebras. In particular, $(H\,\overline{\#}\,H^*)\#H=H\rtimes H^*\ltimes H$ as algebras.*

5 The category of Doi-Hopf modules

Let H be a Hopf algebra over a field k, A an H-comodule algebra and C an H-module coalgebra. A Doi-Hopf module is a k-vector space together with an A-action and a C-coaction satisfying a certain compatibility relation. They were introduced independently by Doi [9] and Koppinen [16], and it turns out that most types of Hopf modules that had been studied before were special cases: Sweedler's Hopf modules [21], Doi's relative Hopf modules [8], Takeuchi's relative Hopf modules [20], Yetter-Drinfeld modules, graded modules and modules graded by a G-set.
Over a quasi-Hopf algebra, the category of relative Hopf modules has been introduced and studied [3], as well as the category of Hopf H-bimodules (see [14]) and the category of Hopf modules ${}^H_H\mathcal{M}^H_H$ (see [19]). We will introduce Doi-Hopf modules, and we will show that, at least in the case where H is finite dimensional, all these categories are isomorphic to certain categories of Doi-Hopf modules.
First we recall from [3] the definition of a relative Hopf module. Let H be a quasi-bialgebra and C a right H-module coalgebra. Let N be a k-vector space furnished with the following additional structure:

- N is a right H-module; the right action of $h\in H$ on $n\in N$ is denoted by nh;
- N is a left C-comodule in the monoidal category $\mathcal{M}_H$; we use the following notation for the left C-coaction on N: $\rho_N:\ N\to C\otimes N$, $\rho_N(n)=\sum n_{[-1]}\otimes n_{[0]}$; this means that the following conditions hold, for all $n\in N$:

$$\begin{aligned}
&\sum\underline{\varepsilon}(n_{[-1]})n_{[0]}=n \\
&(\underline{\Delta}\otimes id_N)(\rho_N(n))\Phi^{-1}=(id_C\otimes\rho_N)(\rho_N(n))
\end{aligned}\tag{5.1}$$

- we have the following compatibility relation, for all $n\in N$ and $c\in C$:

$$\rho_N(nh)=\sum n_{[-1]}\cdot h_1\otimes n_{[0]}h_2. \tag{5.2}$$

Then N is called a left $[C,H]$-Hopf module. ${}^C\mathcal{M}_H$ is the category of left $[C,H]$-Hopf modules; the morphisms are right H-linear maps which are also left C-comodule maps. We will now generalize this definition.

Definition 5.1 *Let H be a quasi-bialgebra over a field k, C a right H-module coalgebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ a left H-comodule algebra. A right-left $(H,\mathfrak{B},C)$-Hopf module (or Doi-Hopf module) is a k-module N, with the following additional structure: N is right $\mathfrak{B}$-module (the right action of $\mathfrak{b}$ on n is denoted by $n\mathfrak{b}$), and we have a k-linear map $\rho_N:\ N\to C\otimes N$, such that the following relations hold, for all $n\in N$ and $\mathfrak{b}\in\mathfrak{B}$:*

$$(\underline{\Delta}\otimes id_N)(\rho_N(n)) = (id_C\otimes\rho_N)(\rho_N(n))\Phi_\lambda \tag{5.3}$$

$$(\underline{\varepsilon}\otimes id_N)(\rho_N(n)) = n \tag{5.4}$$

$$\rho_N(n\mathfrak{b}) = \sum n_{[-1]}\cdot\mathfrak{b}_{[-1]}\otimes n_{[0]}\mathfrak{b}_{[0]}. \tag{5.5}$$

As usual, we use the Sweedler-type notation $\rho_N(n)=\sum n_{[-1]}\otimes n_{[0]}$. ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is the category of right-left $(H,\mathfrak{B},C)$-Hopf modules and right $\mathfrak{B}$-linear, left C-colinear k-linear maps.

Obviously, if $\mathfrak{B}=H$, $\lambda=\Delta$ and $\Phi_\lambda=\Phi$, then ${}^C\mathcal{M}(H)_{\mathfrak{B}}={}^C\mathcal{M}_H$.
The main aim of Section 6 will be to define the category of two-sided two-cosided Hopf modules over a quasi-bialgebra, and to prove that it is isomorphic to a module category in the finite dimensional case. To this end, we will need our next result, stating that the category of Doi-Hopf modules is a module category in the case where the coalgebra C is finite dimensional. In fact, for an arbitrary right H-module coalgebra C, the linear dual space of C, C^*, is a left H-module algebra. The multiplication of C^* is the convolution, that is $(c^*d^*)(c)=\sum c^*(c_{\underline{1}})d^*(c_{\underline{2}})$, the unit is $\underline{\varepsilon}$ and the left H-module structure is given by $(h\rightharpoonup c^*)(c)=c^*(c\cdot h)$, for $h\in H$, $c^*,d^*\in C^*$, $c\in C$. Thus C^* is a left H-module algebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ is a left H-comodule algebra. By Proposition 4.1, it makes sense to consider the generalized smash product algebra $C^*\blacktriangleright\!\!<\mathfrak{B}$.

Proposition 5.2 *Let H be a quasi-bialgebra, C a finite dimensional right H-module coalgebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ a left H-comodule algebra. Then the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ of right-left $(H,\mathfrak{B},C)$-Hopf modules is isomorphic to the category $\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}}$ of right modules over $C^*\blacktriangleright\!\!<\mathfrak{B}$.*

Proof. We restrict to defining the functors that define the isomorphism of categories, leaving all other details to the reader. Let $\{c_i\}_{i=\overline{1,n}}$ and $\{c^i\}_{i=\overline{1,n}}$ be dual bases in C and C^*.
Let N be a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. Since $i:\ \mathfrak{B}\to C^*\blacktriangleright\!\!<\mathfrak{B}$, $i(\mathfrak{b})=\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}$ for $\mathfrak{b}\in\mathfrak{B}$, is an algebra map, it follows that N is a right $\mathfrak{B}$-module via the action $n\mathfrak{b}=ni(\mathfrak{b})=n(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})$, $n\in N$, $\mathfrak{b}\in\mathfrak{B}$. The map $j:\ C^*\to C^*\blacktriangleright\!\!<\mathfrak{B}$, $j(c^*)=c^*\blacktriangleright\!\!<1_{\mathfrak{B}}$, $c^*\in C^*$, is not an algebra map (it is not multiplicative) but it can be used to define a left C-coaction on N:

$$\rho_N(n)=\sum n_{[-1]}\otimes n_{[-1]}=\sum_{i=1}^n c_i\otimes nj(c^*)=\sum_{i=1}^n c_i\otimes n(c^i\blacktriangleright\!\!<1_{\mathfrak{B}}). \tag{5.6}$$

We can easily check that N becomes an object in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$.
Conversely, take $N\in{}^C\mathcal{M}(H)_{\mathfrak{B}}$. Then N is a right $\mathfrak{B}$-module and C^* acts on M from the right as follows: let $nc^*=\sum c^*(n_{-1]})n_{[0]}$, $n\in N$, $c^*\in C^*$. Now define

$$n(c^*\blacktriangleright\!\!<\mathfrak{b})=(nc^*)\mathfrak{b}=\sum c^*(n_{[-1]})n_{[0]}\mathfrak{b}. \tag{5.7}$$

Then N becomes a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. □

6 Two-sided two-cosided Hopf modules

Now we define the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$. If H is finite dimensional, then this category is isomorphic to a certain category of right-left Doi-Hopf modules, ${}^C\mathcal{M}(H\otimes$

$H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$. As a consequence, if C is also finite dimensional then this category is isomorphic to the category of right modules over a generalized smash product, by Proposition 5.2.

Definition 6.1 *[12, Definition 8.2]. Let H be a quasi-bialgebra. An H-bicomodule algebra $\mathbb{A}$ is a quintuple $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$, where λ and ρ are left and right H-coactions on $\mathbb{A}$, and where $\Phi_\lambda \in H\otimes H\otimes\mathbb{A}$, $\Phi_\rho\in\mathbb{A}\otimes H\otimes H$ and $\Phi_{\lambda,\rho}\in H\otimes\mathbb{A}\otimes H$ are invertible elements, such that*

- *$(\mathbb{A},\lambda,\Phi_\lambda)$ is a left H-comodule algebra,*
- *$(\mathbb{A},\rho,\Phi_\rho)$ is a right H-comodule algebra,*
- *the following compatibility relations hold, for all $a\in\mathbb{A}$:*

$$\Phi_{\lambda,\rho}(\lambda\otimes id)(\rho(a)) = (id\otimes\rho)(\lambda(a))\Phi_{\lambda,\rho} \tag{6.1}$$

$$(1_H\otimes\Phi_{\lambda,\rho})(id\otimes\lambda\otimes id)(\Phi_{\lambda,\rho})(\Phi_\lambda\otimes 1_H) = (id\otimes id\otimes\rho)(\Phi_\lambda)(\Delta\otimes id\otimes id)(\Phi_{\lambda,\rho}) \tag{6.2}$$

$$(1_H\otimes\Phi_\rho)(id\otimes\rho\otimes id)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho}\otimes 1_H) = (id\otimes id\otimes\Delta)(\Phi_{\lambda,\rho})(\lambda\otimes id\otimes id)(\Phi_\rho). \tag{6.3}$$

It was pointed out in [12] that the following additional relations hold in an H-bicomodule algebra $\mathbb{A}$:

$$(id_H\otimes id_{\mathbb{A}}\otimes\varepsilon)(\Phi_{\lambda,\rho}) = 1_H\otimes 1_{\mathbb{A}},\ \ (\varepsilon\otimes id_{\mathbb{A}}\otimes id_H)(\Phi_{\lambda,\rho}) = 1_{\mathbb{A}}\otimes 1_H. \tag{6.4}$$

As a first example, take $\mathbb{A}=H$, $\lambda=\rho=\Delta$ and $\Phi_\lambda=\Phi_\rho=\Phi_{\lambda,\rho}=\Phi$. Related to the left and right comodule algebra structures of $\mathbb{A}$ we will keep the notation of the previous Sections. We will use the following notation:

$$\Phi_{\lambda,\rho} = \sum\Omega^1\otimes\Omega^2\otimes\Omega^3 = \sum\overline{\Omega}^1\otimes\overline{\Omega}^2\otimes\overline{\Omega}^3 = \text{etc.}$$

and

$$\Phi_{\lambda,\rho}^{-1} = \sum\omega^1\otimes\omega^2\otimes\omega^3 = \sum\overline{\omega}^1\otimes\overline{\omega}^2\otimes\overline{\omega}^3 = \text{etc.}$$

If H is a quasi-bialgebra, then the opposite algebra $H^{\rm op}$ is also a quasi-bialgebra. The reassociator of $H^{\rm op}$ is $\Phi_{\rm op}=\Phi^{-1}$. $H\otimes H^{\rm op}$ is also a quasi-bialgebra with reassociator

$$\Phi_{H\otimes H^{\rm op}} = \sum(X^1\otimes x^1)\otimes(X^2\otimes x^2)\otimes(X^3\otimes x^3). \tag{6.5}$$

If we identify $H\otimes H^{\rm op}$-modules and (H,H)-bimodules, then the category of (H,H)-bimodules, ${}_H\mathcal{M}_H$, is monoidal. The associativity constraints are given by $\mathbf{a}'_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$, where

$$\mathbf{a}'_{U,V,W}((u\otimes v)\otimes w) = \Phi\cdot(u\otimes(v\otimes w))\cdot\Phi^{-1} \tag{6.6}$$

for all $U,V,W\in{}_H\mathcal{M}_H$, $u\in U$, $v\in V$ and $w\in W$. A coalgebra in the category of (H,H)-bimodules will be called an H-bimodule coalgebra. More precisely, an H-bimodule coalgebra C is an (H,H)-bimodule (denote the actions by $h\cdot c$ and $c\cdot h$) with a comultiplication $\underline{\Delta}:\ C\to C\otimes C$ and a counit $\underline{\varepsilon}:\ C\to k$ satisfying the following relations, for all $c\in C$ and $h\in H$:

$$\Phi\cdot(\underline{\Delta}\otimes id)(\underline{\Delta}(c))\cdot\Phi^{-1} = (id\otimes\underline{\Delta})(\underline{\Delta}(c)) \tag{6.7}$$

$$\underline{\Delta}(h\cdot c) = \sum h_1\cdot c_{\underline{1}}\otimes h_2\cdot c_{\underline{2}},\ \ \underline{\Delta}(c\cdot h) = \sum c_{\underline{1}}\cdot h_1\otimes c_{\underline{2}}\cdot h_2 \tag{6.8}$$

$$\underline{\varepsilon}(h\cdot c) = \varepsilon(h)\underline{\varepsilon}(c),\ \ \underline{\varepsilon}(c\cdot h) = \underline{\varepsilon}(c)\varepsilon(h) \tag{6.9}$$

where we used the same Sweedler-type notation as before. An H-bimodule coalgebra C becomes a right $H\otimes H^{\rm op}$-module coalgebra via the right $H\otimes H^{\rm op}$-action

$$c\cdot(h\otimes h') = h'\cdot c\cdot h \tag{6.10}$$

for $c\in C$ and $h,h'\in H$. Our next definition extends the definition of two-sided two-cosided Hopf modules from [19].

Definition 6.2 *Let H be a quasi-bialgebra, $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ an H-bicomodule algebra and C an H-bimodule coalgebra. A two-sided two-cosided $(H,\mathbb{A},C)$-Hopf module (or crossed Hopf module) is a k-vector space with the following additional structure:*

- *N is an $(H,\mathbb{A})$-two-sided Hopf module, i.e. $N\in {}_H\mathcal{M}_{\mathbb{A}}^H$; we write $\succ$ for the left H-action, $\prec$ for the right $\mathbb{A}$-action, and $\rho_N^H(n)=\sum n_{(0)}\otimes n_{(1)}$ for the right H-coaction on $n\in N$;*
- *we have k-linear map $\rho_N^C:\ N\to C\otimes N$, $\rho_N^C(n)=\sum n_{[-1]}\otimes n_{[0]}$, called the left C-coaction on N, such that $\sum\underline{\varepsilon}(n_{[-1]})n_{[0]}=n$ and*

$$\Phi(\underline{\Delta}\otimes id_N)(\rho_N^C(n))=(id_C\otimes\rho_N^C)(\rho_N^C(n))\Phi_\lambda \tag{6.11}$$

  for all $n\in N$;
- *N is a (C,H)-"bicomodule", in the sense that, for all $n\in N$,*

$$\Phi(\rho_N^C\otimes id_H)(\rho_N^H(n))=(id_C\otimes\rho_N^H)(\rho_N^C(n))\Phi_{\lambda,\rho} \tag{6.12}$$

- *the following compatibility relations hold*

$$\rho_N^C(h\succ n) = \sum h_1\cdot n_{[-1]}\otimes h_2\succ n_{[0]} \tag{6.13}$$
$$\rho_N^H(n)(n\prec a) = \sum n_{[-1]}\cdot a_{[-1]}\otimes n_{[0]}\prec a_{[0]} \tag{6.14}$$

  for all $h\in H$, $n\in N$ and $a\in\mathbb{A}$.

${}_H^C\mathcal{M}_{\mathbb{A}}^H$ will be the category of two-sided two-cosided Hopf modules and maps preserving the actions by H and $\mathbb{A}$ and the coactions by H and C.

Let H be a quasi-bialgebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Let us call the threetuple $(H,\mathbb{A},C)$ a *Drinfeld datum.* In the rest of this Section we will show that if H is a finite dimensional quasi-Hopf algebra then the above category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to a certain category of Doi-Hopf modules. To this end, we first need some lemmas.

Lemma 6.3 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ an H-bicomodule algebra. Consider the map*

$$\wp:\ (\mathbb{A}\,\overline{\#}\,H^*)\#H\to(H\otimes H^{\mathrm{op}})\otimes(\mathbb{A}\,\overline{\#}\,H^*)\#H$$

given by

$$\wp((a\,\overline{\#}\,\varphi)\#h)=\sum a_{[-1]}\omega^1\otimes S(y^3h_2)\otimes(a_{[0]}\omega^2\,\overline{\#}\,y^1\rightharpoonup\varphi\leftharpoonup\omega^3)\#y^2h_1 \tag{6.15}$$

for any $a\in\mathbb{A}$, $\varphi\in H^*$ and $h\in H$, where $\Phi_{\lambda,\rho}^{-1}=\sum\omega^1\otimes\omega^2\otimes\omega^3$. Set

$$\Phi_\wp=\sum(\tilde{X}_\lambda^1\otimes g^1S(x^3))\otimes(\tilde{X}_\lambda^2\otimes g^2S(x^2))\otimes(\tilde{X}_\lambda^3\,\overline{\#}\,\varepsilon)\#x^1 \tag{6.16}$$

where $f^{-1}=\sum g^1\otimes g^2$ is the element defined in (1.16). Then $((\mathbb{A}\,\overline{\#}\,H^*)\#H,\wp,\Phi_\wp)$ is a left $H\otimes H^{\mathrm{op}}$-comodule algebra.

Proof. We first show that $\wp$ is an algebra map. Using (1.30) and (2.11) we can easily show that the multiplication on $(\mathbb{A}\,\overline{\#}\,H^*)\#H$ is given by

$$\begin{aligned}&((a\,\overline{\#}\,\varphi)\#h)((a'\,\overline{\#}\,\psi)\#h')\\ &\quad= \sum aa'_{<0>}\tilde{x}^1_\rho\,\overline{\#}\,(x^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\tilde{x}^2_\rho)(x^2h_1\rightharpoonup\psi\leftharpoonup\tilde{x}^3_\rho)\#x^3h_2h' \end{aligned}\tag{6.17}$$

for all $a,a'\in\mathbb{A}$, $\varphi,\psi\in H^*$ and $h,h'\in H$. Therefore

$$\begin{aligned}&\wp(((a\,\overline{\#}\,\varphi)\#h)((a'\,\overline{\#}\,\psi)\#h'))\\ &\quad= \sum a_{[-1]}a'_{<0>_{[-1]}}(\tilde{x}^1_\rho)_{[-1]}\omega^1\otimes S(y^3x^3_2h_{(2,2)}h'_2)\otimes[a_{[0]}a'_{<0>_{[0]}}(\tilde{x}^1_\rho)_{[0]}\omega^2\\ &\qquad\overline{\#}\,(y^1_1x^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\tilde{x}^2_\rho\omega^3_1)(y^1_2x^2h_1\rightharpoonup\psi\leftharpoonup\tilde{x}^3_\rho\omega^3_2)]\#y^2x^3_1h_{(2,1)}h'_1\\ (6.3,1.3)&\quad= \sum a_{[-1]}a'_{<0>_{[-1]}}\overline{\omega}^1\omega^1\otimes S(y^3x^3h_{(2,2)}h'_2)\otimes[a_{[0]}a'_{<0>_{[0]}}\overline{\omega}^2\omega^2_{<0>}\tilde{x}^1_\rho\\ &\qquad\overline{\#}\,(z^1y^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\overline{\omega}^3\omega^2_{<1>}\tilde{x}^2_\rho)(z^2y^2_1x^1h_1\rightharpoonup\psi\leftharpoonup\omega^3\tilde{x}^3_\rho)]\#z^3y^2_2x^2h_{(2,1)}h'_1\\ (6.1,1.1)&\quad= \sum a_{[-1]}\overline{\omega}^1a'_{[-1]}\omega^1\otimes S(y^3h_2)\cdot_{op}S(x^3h'_2)\otimes[a_{[0]}\overline{\omega}^2(a'_{[0]}\omega^2)_{<0>}\tilde{x}^1_\rho\,\overline{\#}\,(z^1y^1\\ &\qquad\rightharpoonup\varphi\leftharpoonup\overline{\omega}^3(a'_{[0]}\omega^2)_{<1>}\tilde{x}^2_\rho)(z^2y^2_1h_{(1,1)}x^1\rightharpoonup\psi\leftharpoonup\omega^3\tilde{x}^3_\rho)]\#z^3y^2_2h_{(1,2)}x^2h'_1\\ (2.11)&\quad= \sum a_{[-1]}\overline{\omega}^1a'_{[-1]}\omega^1\otimes S(y^3h_2)\cdot_{op}S(x^3h'_2)\otimes[(a_{[0]}\overline{\omega}^2\,\overline{\#}\,z^1y^1\rightharpoonup\varphi\leftharpoonup\overline{\omega}^3)\\ &\qquad(a'_{[0]}\omega^2\,\overline{\#}\,z^2y^2_1h_{(1,1)}x^1\rightharpoonup\psi\leftharpoonup\omega^3)]\#z^3y^2_2h_{(1,2)}x^2h'_1\\ (1.30)&\quad= \sum a_{[-1]}\overline{\omega}^1a'_{[-1]}\omega^1\otimes S(y^3h_2)\cdot_{op}S(x^3h'_2)\otimes[(a_{[0]}\overline{\omega}^2\,\overline{\#}\,y^1\rightharpoonup\varphi\leftharpoonup\overline{\omega}^3)\#y^2h_1]\\ &\qquad[(a'_{[0]}\omega^2\,\overline{\#}\,x^1\rightharpoonup\psi\leftharpoonup\omega^3)\#x^2h'_1]\\ &\quad= \wp((a\,\overline{\#}\,\varphi)\#h)\wp((a'\,\overline{\#}\,\psi)\#h')\end{aligned}$$

where $\cdot_{\mathrm{op}}$ is the product in H^{op}. Obviously $\wp$ respects the unit element and (2.7) holds. (2.5) can be proved using similar computations as above and is left to the reader. Using the notation

$$\Phi_\wp=\sum\tilde{X}^1_\wp\otimes\tilde{X}^2_\wp\otimes\tilde{X}^3_\wp=\text{etc.}$$

we can compute

$$\begin{aligned}&(id\otimes id\otimes\wp)(\Phi_\wp)(\Delta\otimes id\otimes id)(\Phi\wp)\\ &\quad= \sum(\tilde{X}^1_\lambda\otimes g^1S(x^3))((\tilde{Y}^1_\lambda)_1\otimes G^1_1S(y^3)_1)\otimes(\tilde{X}^2_\lambda\otimes g^2S(x^2))\\ &\qquad((\tilde{Y}^1_\lambda)_2\otimes G^1_2S(y^3)_2)\otimes((\tilde{X}^3_\lambda)_{[-1]}\otimes S(x^1_2))(\tilde{Y}^2_\lambda\otimes G^2S(y^2))\\ &\qquad\otimes[((\tilde{X}^3_\lambda)_{[0]}\,\overline{\#}\,\varepsilon)\#x^1_1][(\tilde{Y}^3_\lambda\,\overline{\#}\,\varepsilon)\#y^1]\\ (1.11,1.3)&\quad= \sum(\tilde{X}^1_\lambda(\tilde{Y}^1_\lambda)_1\otimes G^1_1g^1S(y^3x^3))\otimes(\tilde{X}^2_\lambda(\tilde{Y}^1_\lambda)_2\otimes G^1_2g^2S(z^3y^2_2x^2))\\ &\qquad\otimes((\tilde{X}^3_\lambda)_{[-1]}\tilde{Y}^2_\lambda\otimes G^2S(z^2y^2_1x^1))\otimes[((\tilde{X}^3_\lambda)_{[0]}\tilde{Y}^3_\lambda\,\overline{\#}\,\varepsilon)\#z^1y^1]\\ (2.6,1.9,1.18)&\quad= \sum(\tilde{Y}^1_\lambda X^1\otimes x^1g^1S(y^3))\otimes(\tilde{X}^1_\lambda(\tilde{Y}^2_\lambda)_1X^2\otimes x^2g^2_1G^1S(z^3y^2_2))\\ &\qquad\otimes(\tilde{X}^2_\lambda(\tilde{Y}^2_\lambda)_2X^3\otimes x^3g^2_2G^2S(z^2y^2_1))\otimes[(\tilde{X}^3_\lambda\tilde{Y}^3_\lambda\,\overline{\#}\,\varepsilon)\#z^1y^1]\\ (1.11)&\quad= \sum(\tilde{Y}^1_\lambda\otimes g^1S(y^3))(X^1\otimes x^1)\otimes(\tilde{X}^1_\lambda\otimes G^1S(z^3))((\tilde{Y}^2_\lambda)_1\otimes g^2_1S(y^2)_1)\\ &\qquad(X^2\otimes x^2)\otimes(\tilde{X}^2_\lambda\otimes G^2S(z^2))((\tilde{Y}^2_\lambda)_2\otimes g^2_2S(y^2)_2)(X^3\otimes x^3)\\ &\qquad\otimes[(\tilde{X}^3_\lambda\,\overline{\#}\,\varepsilon)\#z^1][(\tilde{Y}^3_\lambda\,\overline{\#}\,\varepsilon)\#y^1]\\ (6.5)&\quad= (1_H\otimes\Phi_\wp)(id\otimes\Delta_{H\otimes H^{\mathrm{op}}}\otimes id)(\Phi_\wp)(\Phi_{H\otimes H^{\mathrm{op}}}\otimes\mathbf{1})\end{aligned}$$

where $\sum G^1 \otimes G^2$ is another copy of f^{-1} and $\mathbf{1} = (1_{\mathbb{A}} \,\overline{\#}\, \varepsilon)\#1_H$ is the unit of the algebra $(\mathbb{A} \,\overline{\#}\, H^*)\#H$. $\square$

Let H be a finite dimensional quasi-Hopf algebra, $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra and C an H-bimodule coalgebra. By Lemma 6.3, we can consider the category of Doi-Hopf modules ${}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A} \,\overline{\#}\, H^*)\#H}$. We will prove that it is isomorphic to the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$.

Lemma 6.4 *Let H be a quasi-Hopf algebra and $(H, \mathbb{A}, C)$ a Drinfeld datum. We have a functor*

$$F : {}^C_H\mathcal{M}^H_{\mathbb{A}} \to {}^C\mathcal{M}(H \otimes H^{\mathrm{op}})_{(\mathbb{A} \,\overline{\#}\, H^*)\#H}.$$

$F(N) = N$ as a k-module, with structure maps given by the equations

$$n \leftarrow ((a \,\overline{\#}\, \varphi)\#h) = \sum \varphi S^{-1}(f^2 n_{(1)} a_{<1>} \tilde{p}^2_\rho) S(h) f^1 \succ n_{(0)} \prec a_{<0>} \tilde{p}^1_\rho \tag{6.18}$$

$$\tilde{\rho}^C_N(n) = \sum n_{\{-1\}} \otimes n_{\{0\}} = \sum f^1 \cdot n_{[-1]} \otimes f^2 \succ n_{[0]} \tag{6.19}$$

for all $n \in N$, $a \in \mathbb{A}$, $\varphi \in H^*$ and $h \in H$. F sends a morphism to itself.

Proof. Since N is a two-sided $(\mathbb{A}, H)$-Hopf module, we know by (3.38) that N is a right $(\mathbb{A} \,\overline{\#}\, H^*)\#H$-module via the action defined by (6.18). Let $\sum F^1 \otimes F^2$ be another copy of f. For any $n \in N$, we have that

$$\begin{aligned}
& (\underline{\Delta} \otimes id_N)(\tilde{\rho}^C_N(n))\Phi^{-1}_{\wp} \\
(6.16) \quad &= \sum n_{\{-1\}_{\underline{1}}} \cdot (\tilde{x}^1_\lambda \otimes S(X^3)F^1) \otimes n_{\{-1\}_{\underline{2}}} \cdot (\tilde{x}^2_\lambda \otimes S(X^2)F^2) \\
& \quad \otimes n_{\{0\}} \leftarrow [(\tilde{x}^3_\lambda \,\overline{\#}\, \varepsilon)\#X^1] \\
(6.10, 6.19) \quad &= \sum S(X^3)F^1 \cdot (f^1 \cdot n_{[-1]})_{\underline{1}} \cdot \tilde{x}^1_\lambda \otimes S(X^2)F^2 \cdot (f^1 \cdot n_{[-1]})_{\underline{2}} \cdot \tilde{x}^2_\lambda \\
& \quad \otimes S(X^1) f^2 \succ n_{[0]} \prec \tilde{x}^3_\lambda \\
(6.8) \quad &= \sum S(X^3)F^1 f^1_1 \cdot n_{[-1]_{\underline{1}}} \cdot \tilde{x}^1_\lambda \otimes S(X^2)F^2 f^1_2 \cdot n_{[-1]_{\underline{2}}} \cdot \tilde{x}^2_\lambda \\
& \quad \otimes S(X^1) f^2 \succ n_{[0]} \prec \tilde{x}^3_\lambda \\
(6.11, 1.9, 1.18) \quad &= \sum f^1 \cdot n_{[-1]} \otimes F^1 f^2_1 \cdot n_{[0,-1]} \otimes F^2 f^2_2 \succ n_{[0,0]} \\
(6.13) \quad &= \sum f^1 \cdot n_{[-1]} \otimes F^1 \cdot (f^2 \succ n_{[0]})_{[-1]} \otimes F^2 \succ (f^2 \succ n_{[0]})_{[0]} \\
(6.19) \quad &= \sum n_{\{-1\}} \otimes F^1 \cdot n_{\{0\}_{[-1]}} \otimes F^2 \succ n_{\{0\}_{[0]}} \\
(6.19) \quad &= (id_C \otimes \tilde{\rho}^C_N)(\tilde{\rho}^C_N(n)).
\end{aligned}$$

We still have to show the compatibility relation (5.5). First observe that (3.6), (6.3) and (1.5) imply

$$\sum \Omega^1(\tilde{p}^1_\rho)_{[-1]} \otimes \Omega^2(\tilde{p}^1_\rho)_{[0]} \otimes \Omega^3 \tilde{p}^2_\rho = \sum \omega^1 \otimes \omega^2_{<0>} \tilde{p}^1_\rho \otimes \omega^2_{<1>} \tilde{p}^2_\rho S(\omega^3). \tag{6.20}$$

For all $n \in N$, $a \in \mathbb{A}$, $\varphi \in H^*$ and $h \in H$ we compute that

$$\begin{aligned}
& \tilde{\rho}^C_N(n \leftarrow ((a \,\overline{\#}\, \varphi)\#h)) \\
(6.18, 6.19) \quad &= \sum \varphi S^{-1}(f^2 n_{(1)} a_{<1>} \tilde{p}^2_\rho) F^1 \cdot (S(h) f^1 \succ n_{(0)} \prec a_{<0>} \tilde{p}^1_\rho)_{[-1]} \\
& \quad \otimes F^2 \succ (S(h) f^1 \succ n_{(0)} \prec a_{<0>} \tilde{p}^1_\rho)_{[0]} \\
(6.13, 1.11) \quad &= \sum \varphi S^{-1}(f^2 n_{(1)} a_{<1>} \tilde{p}^2_\rho) S(h_2) F^1 f^1_1 \cdot n_{(0)_{[-1]}} \cdot a_{<0>_{[-1]}} (\tilde{p}^1_\rho)_{[-1]} \\
& \quad \otimes S(h_1) F^2 f^1_2 \succ n_{(0)_{[0]}} \prec a_{<0>_{[0]}} (\tilde{p}^1_\rho)_{[0]}
\end{aligned}$$

$$
\begin{aligned}
(6.1) \quad &= \sum \varphi S^{-1}(f^2 n_{(1)}\omega^3 a_{[0]_{<1>}}\Omega^3\tilde{p}^2_\rho)S(h_2)F^1 f^1_1 \cdot n_{(0)_{[-1]}}\cdot \omega^1 a_{[-1]}\Omega^1(\tilde{p}^1_\rho)_{[-1]} \\
&\quad \otimes S(h_1)F^2 f^1_2 \succ n_{(0)_{[0]}} \prec \omega^2 a_{[0]_{<0>}}\Omega^2(\tilde{p}^1_\rho)_{[0]} \\
(6.20,6.12) \quad &= \sum \varphi S^{-1}(f^2 x^3 n_{[0]_{(1)}} a_{[0]_{<1>}}\omega^2_{<1>}\tilde{p}^2_\rho S(\omega^3))S(h_2)F^1 f^1_1 x^1 \cdot n_{[-1]}\cdot a_{[-1]}\omega^1 \\
&\quad \otimes S(h_1)F^2 f^1_2 x^2 \succ n_{[0]_{(0)}} \prec a_{[0]_{<0>}}\omega^2_{<0>}\tilde{p}^1_\rho \\
(1.9,1.18) \quad &= \sum (x^1 \rightharpoonup \varphi \leftharpoonup \omega^3)S^{-1}(F^2 f^2_2 n_{[0]_{(1)}} a_{[0]_{<1>}}\omega^2_{<1>}\tilde{p}^2_\rho)S(x^3 h_2) f^1 \cdot n_{[-1]}\cdot a_{[-1]}\omega^1 \\
&\quad \otimes S(x^2 h_1)F^1 f^2_1 \succ n_{[0]_{(0)}} \prec a_{[0]_{<0>}}\omega^2_{<0>}\tilde{p}^1_\rho \\
(6.13) \quad &= \sum (x^1 \rightharpoonup \varphi \leftharpoonup \omega^3)S^{-1}(F^2(f^2 \succ n_{[0]})_{(1)} a_{[0]_{<1>}}\omega^2_{<1>}\tilde{p}^2_\rho) \\
&\quad S(x^3 h_2) f^1 \cdot n_{[-1]}\cdot a_{[-1]}\omega^1 \otimes S(x^2 h_1)F^1 \succ (f^2 \succ n_{[0]})_{(0)} \prec a_{[0]_{<0>}}\omega^2_{<0>}\tilde{p}^1_\rho \\
(6.10,6.18) \quad &= \sum f^1 \cdot n_{[-1]}\cdot [a_{[-1]}\omega^1 \otimes S(x^3 h_2)] \\
&\quad \otimes (f^2 \succ n_{[0]}) \leftarrow [(a_{[0]}\omega^2 \mathbin{\overline{\#}} x^1 \rightharpoonup \varphi \leftharpoonup \omega^3)\# x^2 h_1] \\
(6.19,6.15) \quad &= \tilde{\rho}^C_N(n)\wp((a \mathbin{\overline{\#}} \varphi)\# h)
\end{aligned}
$$

completing the proof. □

Lemma 6.5 *Let H be a finite dimensional quasi-Hopf algebra and $(H,\mathbb{A},C)$ a Drinfeld datum. We have a functor*

$$G:\ {}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\mathbin{\overline{\#}} H^*)\# H} \to {}^C_H\mathcal{M}^H_{\mathbb{A}}.$$

$G(N)=N$ as a k-module, with structure maps given by

$$h\succ n = n\leftarrow[(1_{\mathbb{A}}\mathbin{\overline{\#}}\varepsilon)\# S^{-1}(h)],\ \ n\prec a = n\leftarrow[(a\mathbin{\overline{\#}}\varepsilon)\# 1_H], \tag{6.21}$$

$$\rho^H_N:\ N\to N\otimes H,\ \ \rho^H_N(n)=\sum_{i=1}^n n\leftarrow[(\tilde{q}^1_\rho \mathbin{\overline{\#}} S^{-1}(g^2)\rightharpoonup e^i S\leftharpoonup \tilde{q}^2_\rho)\# S^{-1}(g^1)]\otimes e_i, \tag{6.22}$$

$$\underline{\rho}^C_N: N\to C\otimes N,\ \ \underline{\rho}^C_N(n)=\sum g^1\cdot n_{[-1]}\otimes g^2\succ n_{[0]} \tag{6.23}$$

for $n\in N$, $a\in\mathbb{A}$ and $h\in H$. Here $\{e_i\}_{i=\overline{1,n}}$ is a basis of H and $\{e^i\}_{i=\overline{1,n}}$ is the corresponding dual basis of H^*. G sends a morphism to itself.

Proof. Since N is a right $(\mathbb{A}\mathbin{\overline{\#}} H^*)\# H$-module, we already know by (3.35) and (3.37) that H is a two-sided $(\mathbb{A},H)$-Hopf module via (6.21) and (6.22). Thus we only have to check (6.11), (6.12) and (6.13). First note that $N\in {}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\mathbin{\overline{\#}} H^*)\# H}$ implies

$$
\begin{aligned}
&\sum n_{[-1]}\otimes n_{[0,-1]}\otimes n_{[0,0]} \\
&\qquad = \sum S(X^3) f^1\cdot n_{[-1]_{\underline{1}}}\cdot \tilde{x}^1_\lambda \otimes S(X^2) f^2\cdot n_{[-1]_{\underline{2}}}\cdot \tilde{x}^2_\lambda \otimes n_{[0]}\leftarrow[(\tilde{x}^3_\lambda \mathbin{\overline{\#}}\varepsilon)\# X^1] \qquad (6.24)\\
&\sum \{n\leftarrow[(a\mathbin{\overline{\#}}\varphi)\# h]\}_{[-1]}\otimes\{n\leftarrow[(a\mathbin{\overline{\#}}\varepsilon)\# h]\}_{[0]} \\
&\qquad = \sum S(x^3 h_2)\cdot n_{[-1]}\cdot a_{[-1]}\omega^1\otimes n_{[0]}\leftarrow[(a_{[0]}\omega^2\mathbin{\overline{\#}} x^1\rightharpoonup\varphi\leftharpoonup\omega^3)\# x^2 h_1] \qquad (6.25)
\end{aligned}
$$

for all $n\in N$, $a\in\mathbb{A}$, $\varphi\in H^*$ and $h\in H$. By the above definitions and (6.25) it is immediate that

$$\underline{\rho}^C_N(h\succ n)=\Delta(h)\underline{\rho}^C_N(n)\ \text{ and }\ \underline{\rho}^C_N(n\prec a)=\underline{\rho}^C_N(n)\rho_\lambda(a) \tag{6.26}$$

for all $h\in H$, $n\in N$ and $a\in\mathbb{A}$ (we leave it to the reader to verify the details). Let $\sum G^1\otimes G^2$ be another copy of f^{-1}. We compute that

$$\Phi(\underline{\Delta}\otimes id_N)(\rho^C_N(n))$$

$$
\begin{aligned}
(6.23) &= \sum X^1\cdot(g^1\cdot n_{[-1]})_{\underline{1}}\otimes X^2\cdot(g^1\cdot n_{[-1]})_{\underline{2}}\otimes X^3g^2\succ n_{[0]}\\
(6.21,6.8) &= \sum X^1g^1_1\cdot n_{[-1]_{\underline{1}}}\otimes X^2g^1_2\cdot n_{[-1]_{\underline{2}}}\otimes n_{[0]}\leftarrow[(1_{\mathbb{A}}\,\overline{\#}\,\varepsilon)\#S^{-1}(X^3g^2)]\\
(6.24,6.17) &= \sum X^1g^1_1G^1S(x^3)\cdot n_{[-1]}\cdot\tilde{X}^1_\lambda\otimes X^2g^1_2G^2S(x^2)\cdot n_{[0,-1]}\cdot\tilde{X}^2_\lambda\\
&\quad\otimes n_{[0,0]}\leftarrow[(\tilde{X}^3_\lambda\,\overline{\#}\,\varepsilon)\#S^{-1}(X^3g^2S(x^1))]\\
(1.9,1.18) &= \sum g^1\cdot n_{[-1]}\cdot\tilde{X}^1_\lambda\otimes g^2_1G^1\cdot n_{[0,-1]}\cdot\tilde{X}^2_\lambda\\
&\quad\otimes n_{[0,0]}\leftarrow[(\tilde{X}^3_\lambda\,\overline{\#}\,\varepsilon)\#S^{-1}(g^2_2G^2)]\\
(6.21) &= \sum g^1\cdot n_{[-1]}\cdot\tilde{X}^1_\lambda\otimes g^2_1G^1\cdot n_{[0,-1]}\cdot\tilde{X}^2_\lambda\otimes g^2_2G^2\succ n_{[0,0]}\prec\tilde{X}^3_\lambda\\
(6.23,6.8,6.26) &= (id_C\otimes\underline{\rho}^C_N)(\underline{\rho}^C_N(n))\Phi_\lambda.
\end{aligned}
$$

The verification of (6.12) is based on similar computations, and we leave the details to the reader. □

As a consequence of Lemmas 6.4 and 6.5, we have the following description of ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ as a category of Doi-Hopf modules; this description generalizes [1, Proposition 2.3].

Theorem 6.6 *Let H be a finite dimensional quasi-Hopf algebra and $(H,\mathbb{A},C)$ a Drinfeld datum. Then the categories ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ and ${}^C\mathcal{M}(H\otimes H^{\mathrm{op}})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$ are isomorphic.*

Proof. We have to verify that the functors F and G defined in Lemmas 6.4 and 6.5 are inverses. For the C-coactions (6.19) and (6.23), this is obvious; for the other structures, it has been already done in Corollary 3.6. □

Proposition 5.2 and Theorem 6.6 immediately imply the following result.

Corollary 6.7 *Let H be a finite dimensional quasi-Hopf algebra and $(H,\mathbb{A},C)$ a Drinfeld datum with C finite dimensional. Then the category ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to the category of right modules over the generalized smash product $C^*\blacktriangleright\!\!<((\mathbb{A}\,\overline{\#}\,H^*)\#H)$.*

Remark 6.8 Let H be a finite dimensional Hopf algebra. Cibils and Rosso [6] introduced an algebra $X=(H^{\mathrm{op}}\otimes H)\underline{\otimes}(H^*\otimes H^{*op})$ having the property that the category of two-sided two-cosided Hopf modules over H^* coincides with the category of left X-modules. Moreover, it was also proved in [6] that X is isomorphic to the direct tensor product of a Heisenberg double and the opposite of a Drinfeld double. Recently, Panaite [18] introduced two other algebras Y and Z with the same property as X. More precisely, Y is the two-sided crossed product $H^*\#(H\otimes H^{\mathrm{op}})\#H^{*\mathrm{op}}$, and Z is the diagonal crossed product in the sense of [12], $(H^*\otimes H^{*op})\bowtie(H\otimes H^{\mathrm{op}})$. Using different methods, we proved that the category of two-sided two-cosided Hopf modules over a finite dimensional quasi-Hopf algebra is isomorphic to the category of right (resp. left) modules over the generalized smash product $\mathcal{A}=H^*\blacktriangleright\!\!<((H\,\overline{\#}\,H^*)\#H)$ (resp. $\mathcal{A}^{\mathrm{op}}$). Note that, in general, the multiplication on $C^*\blacktriangleright\!\!<((\mathbb{A}\,\overline{\#}\,H^*)\#H)$ is given by the formula

$$
\begin{aligned}
&[c^*\blacktriangleright\!\!<((a\,\overline{\#}\,\varphi)\#h)][d^*\blacktriangleright\!\!<((a'\,\overline{\#}\,\psi)\#h')]\\
&\quad= \sum(\tilde{x}^1_\lambda\rightharpoonup c^*\leftharpoonup S(X^3)f^1)(\tilde{x}^2_\lambda a_{[-1]}\omega^1\rightharpoonup d^*\leftharpoonup S(X^2x^3h_2)f^2)\blacktriangleright\!\!<\{[\tilde{x}^3_\lambda a_{[0]}\omega^2a'_{<0>}\tilde{x}^1_\rho\\
&\qquad\overline{\#}\,(X^1_{(1,1)}y^1x^1\rightharpoonup\varphi\leftharpoonup\omega^3a'_{<1>}\tilde{x}^2_\rho)(X^1_{(1,2)}y^2x^2_1h_{(1,1)}\rightharpoonup\psi\leftharpoonup\tilde{x}^3_\rho)]\#X^1_2y^3x^2_2h_{(1,2)}h'\}
\end{aligned}
$$

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