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BILOCAL DERIVATIONS OF STANDARD OPERATOR ALGEBRAS

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ABSTRACT. In this paper, we shall show the following two results: (1) Let A be a standard operator algebra with I, if Φ is a linear mapping on A which satisfies that $\Phi(T)$ maps $\ker T$ into $\operatorname{ran} T$ for all $T \in A$, then Φ is of the form $\Phi(T) = TA + BT$ for some A, B in $B(X)$. (2) Let X be a Hilbert space, if Φ is a norm-continuous linear mapping on $B(X)$ which satisfies that $\Phi(P)$ maps $\ker P$ into $\operatorname{ran} P$ for all self-adjoint projection P in $B(X)$, then Φ is of the form $\Phi(T) = TA + BT$ for some A, B in $B(X)$.

In what follows X stands for a Banach space (or Hilbert space) and X^* for its norm dual. We denote by (x, f) the duality pairing between elements $f \in X^*$ and $x \in X$, and we use the symbols "$B(X)$", "$L(X)$", "$F(X)$", "I" and "$x \otimes f$" to denote the set of all linear bounded operators on X, the set of all linear mappings on X, the set of all finite rank operators on X, the identity operator and the rank one operator $(*, f)x$ on X, respectively.

If A is a Banach algebra, and A_1 is a Banach subalgebra of A, we say that a linear mapping $\Phi\colon A_1 \to A$ is a derivation if $\Phi(ab) = \Phi(a)b + a\Phi(b)$ for any a and b in A_1. The derivation Φ is called inner if there exists an element a in A such that $\Phi(b) = ba - ab$ for any b in A_1. We say that a linear mapping $\Phi\colon A_1 \to A$ is a local derivation if for every a in A_1, there exists a derivation $\delta_a\colon A_1 \to A$, depending on a, such that $\Phi(a) = \delta_a(a)$. A linear mapping Φ is called a Jordan derivation if $\Phi(a^2) = a\Phi(a) + a\Phi(a)$ for every a in A_1. We give the notion of bilocal derivation as follows:

Definition 1. If A is a Banach subalgebra of $B(X)$, a linear mapping $\Phi\colon A \to B(X)$ is called a bilocal derivation if for every T in A and u in X, there exists a derivation $\delta_{T,u}\colon A \to B(X)$, depending on T and u, such that $\Phi(T)u = \delta_{T,u}(T)u$.

Definition 2. Let X be a Banach space, a Banach subalgebra A of $B(X)$ is called a standard operator algebra if A contains $F(X)$.

D. R. Larson and A. R. Sourour [5] have proved that every local derivation on $B(X)$ is a derivation. R. Kadison [4] and M. Brešar [1] have discussed norm-continuous local derivations on von Neumann algebras. It is obvious that every

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local derivation is bilocal. In this paper, we shall prove that every bilocal derivation $\Phi\colon \mathcal{A} \to B(X)$ is a derivation, where $\mathcal{A}$ is a standard operator algebra.

We now prove the main theorems of this paper.

Theorem 3. *Let $\mathcal{A}$ with I be a standard operator subalgebra of $B(X)$, and let Φ be a linear mapping from $\mathcal{A}$ into $B(X)$, then the following properties hold:*

(I) *If $\Phi(T)$ maps $\ker T$ into $\operatorname{ran} T$ for all T in $\mathcal{A}$, then there exist linear operators A and B in $B(X)$ such that*

$$\Phi(T) = TA + BT, \quad \text{for any } T \text{ in } \mathcal{A}. \tag{1}$$

(II) *If Φ is a bilocal derivation, then Φ is an inner derivation.*

Proof. (I) Fix $f \in X^*$. By the condition (I) of the theorem, $\Phi(x \otimes f)$ maps $\ker f$ into $\operatorname{span}\{x\}$, so there exists a continuous linear functional $\lambda_{f,x}$ on $\ker f$ such that

$$\Phi(x \otimes f)(u) = (u, \lambda_{f,x})x, \quad \text{for any } u \text{ in } \ker f.$$

For any $y \in X$, we have

$$\Phi((x+y) \otimes f)(u) = (u, \lambda_{f,x+y})(x+y). \tag{2}$$

On the other hand,

$$\begin{aligned}\Phi((x+y) \otimes f)(u) &= \Phi(x \otimes f)(u) + \Phi(y \otimes f)(u)\\ &= (u, \lambda_{f,x})x + (u, \lambda_{f,y})y.\end{aligned} \tag{3}$$

By comparing (2) and (3), we see that

$$(u, \lambda_{f,x+y} - \lambda_{f,x})x + (u, \lambda_{f,x+y} - \lambda_{f,y})y = 0.$$

Thus we obtain that $\lambda_{f,x+y} = \lambda_{f,x} = \lambda_{f,y}$, so $\lambda_{f,x}$ is independent of x. We write $\lambda_f = \lambda_{f,x}$. Let g_f be a continuous linear extension of λ_f to X. For any $u \in X - \ker f$, we define an operator $B_{u,f}$ as follows:

$$B_{u,f}x = (u,f)^{-1}(\Phi(x \otimes f)(u) - (u, g_f)x).$$

It is obvious that $B_{u,f} \in L(X)$, thus we see that

$$\Phi(x \otimes f)(u) = (u, g_f)x + (u,f)B_{u,f}x, \quad \text{for any } x \in X. \tag{4}$$

Take any $v \in X - \ker f$ with $v \neq -u$, $v \neq 0$. Thus we have

$$\Phi(x \otimes f)(v) = (v, g_f)x + (v,f)B_{v,f}x, \tag{5}$$

$$\Phi(x \otimes f)(u+v) = (u+v, g_f)x + (u+v, f)B_{u+v,f}x, \quad \text{for any } x \in X. \tag{6}$$

The equalities (4), (5) and (6) yield $B_{u,f} = B_{v,f}$. Thus we may take $B_{u,f} = B_f$. It follows that

$$\Phi(x \otimes f)(u) = (u, g_f)x + (u,f)B_f x, \quad \text{for any } u \text{ in } X. \tag{7}$$

Given any $u \in X$, we now fix a functional $f_0 \in X^*$ with $(u, f_0) = 1$. Thus we may take $g_{f_0} \in X^*$ and $B_{f_0} \in L(X)$ such that

$$\Phi(x \otimes f_0)(u) = (u, g_{f_0})x + (u, f_0)B_{f_0}x, \quad \text{for any } x \in X.$$

We may take $B = B_{f_0}$. For any $f \in X^*$, we write $f_1 = (u,f)f_0$, $f_2 = f - f_1$. It is clear that $(u, f_2) = 0$. Thus there exists a functional g_{f_2} in X^* and B_{f_2} in $L(X)$ such that

$$\Phi(x \otimes f_2)(u) = (u, g_{f_2})x + (u, f_2)B_{f_2}x = (u, g_{f_2})x,$$

for any $x \in X$. If we take $b_{u,f} = (u, f)g_{f_0} + g_{f_2}$, then we have

$$\begin{aligned}\Phi(x \otimes f)(u) &= (\Phi(x \otimes (u, f)f_0) + \Phi(x \otimes f_2))(u)\\ &= (u, (u, f)g_{f_0} + g_{f_2})x + (u, (u, f)f_0)B_{f_0}x\\ &= (x \otimes b_{u,f} + Bx \otimes f)(u).\end{aligned}$$

We define a mapping b_f by $b_f(u) = (u, b_{u,f})x$. Note that $b_f(u) = (u, b_{u,f})x = \Phi(x \otimes f)(u) - (u, f)Bx$ for any x in X, so b_f is a linear functional, and b_f belongs to X^*. It follows that

$$\Phi(x \otimes f)(u) = (x \otimes b_f + Bx \otimes f)(u),$$

for any $u \in X$. Thus we may define an operator C by $Cf = b_f$ for any $f \in X^*$. We claim that $B \in B(X)$ and $C = (\Phi(I) - B)^* \in B(X^*)$. In fact, for any $f \in X^*$ and $x \in X$ with $(x, f) = 1$, if we write $P = x \otimes f$, and note that $\Phi(I - P)$ maps $\ker(I - P)$ ($= \operatorname{span}\{x\}$) into $\operatorname{ran}(I - P)$ ($= \ker f$), then we have $P\Phi(I - P)P = 0$, so $P\Phi(P)P = P\Phi(I)P$. Using the above equality, we can prove $B \in B(X)$ and $C = (\Phi(I) - B)^*$ by imitating the proof of Lemma 5 in [5]. Setting $A = \Phi(I) - B$, thus we have

$$\Phi(x \otimes f) = (x \otimes f)A + B(x \otimes f). \tag{8}$$

Using the same proof as Theorem 1.2 in [5, p. 192], we obtain the desired conclusion.

(II) Since Φ is a bilocal derivation from A into $B(X)$, and every derivation from A into $B(X)$ is inner by [3], Φ satisfies the condition (I) of Theorem 3. Thus there are operators A and B in $B(X)$ such that equality (1) holds, and $A+B = \Phi(I) = 0$, i.e. $B = -A$, so Φ is an inner derivation. The proof is complete. □

Theorem 4. *Let X be a Hilbert space. If Φ is a norm-continuous linear mapping on $B(X)$ which satisfies that $\Phi(P)$ maps $\ker P$ into $\operatorname{ran} P$ for all self-adjoint projection P in $B(X)$, then there are A and B in $B(X)$ such that*

$$\Phi(T) = TA + BT, \quad \text{for any } T \in B(X). \tag{9}$$

Proof. Let P, Q be orthogonal self-adjoint projection in $B(X)$. By the condition of the theorem, we know that $\Phi(P)$ maps $\ker P$ into $\operatorname{ran} P$. Thus we have

$$(I - P)\Phi(P)(I - P) = 0. \tag{10}$$

It follows that $\Phi(P) = \Phi(P)P + P\Phi(P) - P\Phi(P)P$. Substituting $I - P$ for P in the equality (10), we have $P\Phi(I - P)P = 0$, i.e. $P\Phi(P)P = P\Phi(I)P$. Thus we obtain

$$\Phi(P) = \Phi(P)P + P\Phi(P) - P\Phi(I)P. \tag{11}$$

According to the equality (11), we then have

$$\begin{aligned}\Phi(P) + \Phi(Q) &= \Phi(P + Q)\\ &= \Phi(P + Q)(P + Q) + (P + Q)\Phi(P + Q) - (P + Q)\Phi(P + Q)(P + Q)\\ &= (\Phi(P)P + P\Phi(P) - P\Phi(I)P) + (\Phi(Q)Q + Q\Phi(Q) - Q\Phi(I)Q)\\ &\quad + \Phi(P)Q + \Phi(Q)P + P\Phi(Q) + Q\Phi(P) - Q\Phi(I)P - P\Phi(I)Q.\end{aligned}$$

Consequently

$$\Phi(P)Q + \Phi(Q)P + P\Phi(Q) + Q\Phi(P) - Q\Phi(I)P - P\Phi(I)Q = 0. \tag{12}$$

Using the above equality (12), by the same proof as Lemma 1 in [1], we may prove that

$$\Phi(a^2) = \Phi(a)a + a\Phi(a) - a\Phi(I)a, \quad \text{for any } a \text{ in } B(X). \tag{13}$$

We claim that $\Phi(x \otimes y)$ maps $\ker(x \otimes y)$ into $\operatorname{ran}(x \otimes y)$ for every $x, y \in X$. In fact, if $(x, y) \neq 0$, then we have

$$\begin{aligned}\Phi(x \otimes y) &= (x,y)^{-1}\Phi((x \otimes y)^2) \\ &= (x,y)^{-1}\{\Phi(x \otimes y)(x \otimes y) + (x \otimes y)\Phi(x \otimes y) - (x \otimes y)\Phi(I)(x \otimes y)\}.\end{aligned}$$

It follows that $\Phi(x \otimes y)$ maps $\ker(x \otimes y)$ into $\operatorname{span}\{x\}$. If $(x, y) = 0$, without loss of generality, we may assume that $\dim X > 2$ and $x \neq 0, y \neq 0$. Take y_1 and y_2 in X with $(x, y_i) = 1$ and $y_2 \in \{y, y_1\}^\perp$. It is obvious that $\{y, y_1\}^\perp \vee \{y, y_2\}^\perp = \{y\}^\perp = \ker(x \otimes y)$. Since $(x, y_i) = (x, y + y_i) = 1 \neq 0$, and $\Phi(x \otimes y) = \Phi(x \otimes (y + y_i)) - \Phi(x \otimes y_i)$ $(i = 1, 2)$, $\Phi(x \otimes y)$ maps $\{y + y_i\}^\perp \wedge \{y_i\}^\perp$ $(= \{y, y_i\}^\perp)$ into $\operatorname{span}\{x\}$. It follows that $\Phi(x \otimes y)$ maps $\{y, y_1\}^\perp \vee \{y, y_2\}^\perp$ $(= \ker(x \otimes y))$ into $\operatorname{span}\{x\}$.

We may prove that there are A_1 and B_1 in $B(X)$ such that $\Phi(x \otimes y) = (x \otimes y)A_1 + B_1(x \otimes y)$ by imitating the proof of Theorem 3. Thus we have

$$\Phi(F) = FA_1 + B_1F, \quad \text{for any } F \text{ in } F(X). \tag{14}$$

Let Ψ be defined by $\Psi(T) = TA_1 + B_1T$ for every T in $B(X)$, and let $h = \Phi - \Psi$. It is obvious that h satisfies the hypotheses of Φ in the statement of Theorem 4, and $h(F) = 0$ for any F in $F(X)$. Using the equalities (13), we have $Fh(I)F = 0$ for every F in $F(X)$, so $h(I) = 0$. Using the equality (13) to h, we see that h is a Jordan derivation. By Theorem 1 in [2], h is a derivation. It follows by Theorem in [3] that h is an inner derivation. Thus there are A_2 and B_2 in $B(X)$ such that $h(a) = aA_2 + B_2a$ for every a in $B(X)$. Take $A = A_1 + A_2$ and $B = B_1 + B_2$. Thus we obtain that

$$\Phi(a) = h(a) + \Psi(a) = a(A_1 + A_2) + (B_1 + B_2)a = aA + Ba,$$

for any a in $B(X)$. The proof is complete. □

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