

# An Identity Related to Generalized Derivations

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## Abstract

Let  $R$  be a 2-torsion free semiprime ring such that  $R$  has a commutator which is not a zero divisor and  $G: R \rightarrow R$  be an additive mapping such that  $G(xy) = G(x)y + xD(y)$  holds for all  $x, y \in R$  for some derivation  $D$ . Then  $G$  is a generalized derivation.

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## 1 Introduction

This note has been motivated by the work of Molnár [4] and Vukman and Kosi-Ulbl [5]. Throughout,  $R$  will represent an associative ring with center  $Z(R)$ . A ring  $R$  is 2-torsion free, if  $2x = 0$ ,  $x \in R$  implies  $x = 0$ . Recall that  $R$  is prime if  $aRb = (0)$  implies  $a = 0$  or  $b = 0$ , and semiprime if  $aRa = (0)$  implies  $a = 0$ . An additive mapping  $T: R \rightarrow R$  is called a left (right) centralizer in case  $T(xy) = T(x)y$  ( $T(xy) = xT(y)$ ) holds for all  $x, y \in R$ . Zalar [6] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Molnár [4] has proved the following result: Let  $R$  be a 2-torsion free prime ring and let  $T: R \rightarrow R$  be an additive mapping.

If  $T(xy) = T(x)y$  for every  $x, y \in R$ , then  $T$  is a left centralizer. Vukman and Kosi-Ulbl [5] generalized this fact to the semiprime case.

In [2], Hvala has defined the notion of a generalized derivation as follows: An additive mapping  $G : R \rightarrow R$  is said to be a generalized derivation if there exists a derivation  $D : R \rightarrow R$  such that

$$G(xy) = G(x)y + xD(y) \text{ for all } x, y \in R.$$

Also, he calls the map of the form  $x \rightarrow ax + xb$ , where  $a, b$  are fixed elements in  $R$ , an inner generalized derivation. In [3] the authors have defined the notion of Jordan generalized derivations as follows: An additive mapping  $G : R \rightarrow R$  is said to be a Jordan generalized derivation if there exists a derivation  $D : R \rightarrow R$  such that

$$G(x^2) = G(x)x + xD(x) \text{ for all } x \in R.$$

Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map  $f$  satisfying  $f(xy) = f(x)y$  for all  $x, y \in R$ ) and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizer (i.e., an additive map  $f$  satisfying  $f(x^2) = f(x)x$  for all  $x \in R$ ). In [1, Remark 1] Brešar proved the following: For a semiprime ring  $R$ , if  $G$  is a function from  $R$  to  $R$  and  $D : R \rightarrow R$  is an additive mapping such that  $G(xy) = G(x)y + xD(y)$  for all  $x, y \in R$ , then  $D$  is uniquely determined by  $G$  and moreover  $G$  must be a derivation. Moreover, Ashraf and Nadeem-Ur-Rehman [3], proved the following result.

**Theorem 1.1 ([3], Theorem PP. 7)** *Let  $R$  be a 2-torsion free ring such that  $R$  has a commutator which is not a zero divisor. Then every Jordan generalized derivation on  $R$  is a generalized derivation.*

## 2 The Main Result

Our main result is the following theorem.

**Theorem 2.1** *Let  $R$  be a 2-torsion free semiprime ring and let  $G : R \rightarrow R$  be an additive mapping. If  $G(xy) = G(x)y + xD(y)$  for all  $x, y \in R$  for some derivation  $D$  of  $R$ . Then  $G$  is a Jordan generalized derivation.*

**Proof.** The linearizing of the relation

$$G(xy) = G(x)y + xD(y), \quad x, y \in R. \quad (1)$$

gives

$$G(xyz + zyx) = G(x)yz + xD(yz) + G(z)yx + zD(yx), \quad x, y, z \in R. \quad (2)$$

Replacing  $z = x^2$  in (2), we get

$$G(xy x^2 + x^2 yx) = G(x)yx^2 + xD(yx^2) + G(x^2)yx + x^2D(yx), \quad x, y \in R. \quad (3)$$

On the other hand the substitution  $xy + yx$  for  $y$  in the relation (1) gives

$$G(x^2yx + xyx^2) = G(x)xyx + G(x)yx^2 + xD(xy x) + xD(yx^2), \quad x, y \in R. \quad (4)$$

From (3) and (4), we obtain

$$A(x)yx = 0 \quad x, y \in R, \quad (5)$$

where  $A(x)$  stands for  $G(x^2) - G(x)x - xD(x)$ . We intend to prove that

$$A(x) = 0, \quad x \in R. \quad (6)$$

In relation (5) if we replace  $y$  by  $xyA(x)$  then we get,  $A(x)xyA(x)x = 0$ , for all  $x \in R$ , and so, by the semiprimeness of  $R$ , we have

$$A(x)x = 0, \quad x \in R. \quad (7)$$

Multiplying the relation (5) from the left side by  $x$  and from the right by  $A(x)$ , we obtain  $xA(x)yxA(x) = 0$  for all  $x, y \in R$ . By the semiprimeness of  $R$  it follows that

$$xA(x) = 0, \quad x \in R. \quad (8)$$

The linearization of the relation (7) gives

$$A(x)y + \beta(x, y)x + A(y)x + \beta(x, y)y = 0, \quad x, y \in R. \quad (9)$$

where  $\beta(x, y)$  denotes  $G(xy + yx) - G(x)y - G(y)x - xD(y) - yD(x)$ . Putting in (9)  $-x$  for  $x$  we get

$$A(x)y + \beta(x, y)x - A(y)x - \beta(x, y)y = 0, \quad x, y \in R. \quad (10)$$

Adding (9) to (10) and since  $R$  is a 2-torsion free we get

$$A(x)y + \beta(x, y)x = 0, \quad x, y \in R.$$

Right multiplication of the above equation by  $A(x)$  gives  $A(x)yA(x) = 0$ , for all  $x, y \in R$ . Since  $R$  is semiprime, we get  $A(x) = 0$ , for all  $x \in R$ . We have therefore proved that  $G(x^2) = G(x)x + xD(x)$  holds for all  $x \in R$ . In other words,  $G$  is a Jordan generalized derivation and the proof of the theorem is complete.

Now in view of Theorem 1.1 we obtain

**Corollary 2.2** *Let  $R$  be a 2-torsion free semiprime ring such that  $R$  has a commutator which is not a zero divisor and let  $G: R \rightarrow R$  be an additive mapping. If  $G(xy) = G(x)y + xD(y)$  for all  $x, y \in R$  and for a derivation  $D$  of  $R$ , then  $G$  is a generalized derivation.*

It is clear that if we used the derivation  $D$  to be the zero derivation on Theorem 2.1 we get

**Corollary 2.3** *Let  $R$  be a 2-torsion free semiprime ring and let  $T: R \rightarrow R$  be an additive mapping. If  $T(xy) = T(x)y$  for all  $x, y \in R$ , then  $T$  is a left centralizer.*

Now if  $R$  is a prime ring then we get the result of Molnár [4].

**Corollary 2.4** *Let  $R$  be a 2-torsion free prime ring and let  $T: R \rightarrow R$  be an additive mapping. If  $T(xy) = T(x)y$  for all  $x, y \in R$ , then  $T$  is a left centralizer.*

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