

An Identity Related to Generalized Derivations

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Abstract

Let R be a 2-torsion free semiprime ring such that R has a commutator which is not a zero divisor and $G: R \rightarrow R$ be an additive mapping such that $G(xy) = G(x)y + xD(y)$ holds for all $x, y \in R$ for some derivation D . Then G is a generalized derivation.

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1 Introduction

This note has been motivated by the work of Molnár [4] and Vukman and Kosi-Ulbl [5]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is 2-torsion free, if $2x = 0$, $x \in R$ implies $x = 0$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $T: R \rightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. Zalar [6] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Molnár [4] has proved the following result: Let R be a 2-torsion free prime ring and let $T: R \rightarrow R$ be an additive mapping.

If $T(xy) = T(x)y$ for every $x, y \in R$, then T is a left centralizer. Vukman and Kosi-Ulbl [5] generalized this fact to the semiprime case.

In [2], Hvala has defined the notion of a generalized derivation as follows: An additive mapping $G : R \rightarrow R$ is said to be a generalized derivation if there exists a derivation $D : R \rightarrow R$ such that

$$G(xy) = G(x)y + xD(y) \text{ for all } x, y \in R.$$

Also, he calls the map of the form $x \rightarrow ax + xb$, where a, b are fixed elements in R , an inner generalized derivation. In [3] the authors have defined the notion of Jordan generalized derivations as follows: An additive mapping $G : R \rightarrow R$ is said to be a Jordan generalized derivation if there exists a derivation $D : R \rightarrow R$ such that

$$G(x^2) = G(x)x + xD(x) \text{ for all } x \in R.$$

Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in R$) and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizer (i.e., an additive map f satisfying $f(x^2) = f(x)x$ for all $x \in R$). In [1, Remark 1] Brešar proved the following: For a semiprime ring R , if G is a function from R to R and $D : R \rightarrow R$ is an additive mapping such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$, then D is uniquely determined by G and moreover G must be a derivation. Moreover, Ashraf and Nadeem-Ur-Rehman [3], proved the following result.

Theorem 1.1 ([3], **Theorem PP. 7**) *Let R be a 2-torsion free ring such that R has a commutator which is not a zero divisor. Then every Jordan generalized derivation on R is a generalized derivation.*

2 The Main Result

Our main result is the following theorem.

Theorem 2.1 *Let R be a 2-torsion free semiprime ring and let $G : R \rightarrow R$ be an additive mapping. If $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$ for some derivation D of R . Then G is a Jordan generalized derivation.*

Proof. The linearizing of the relation

$$G(xy) = G(x)y + xD(y), \quad x, y \in R. \quad (1)$$

gives

$$G(xyz + zyx) = G(x)yz + xD(yz) + G(z)yx + zD(yx), \quad x, y, z \in R. \quad (2)$$

Replacing $z = x^2$ in (2), we get

$$G(xy x^2 + x^2 yx) = G(x)yx^2 + xD(yx^2) + G(x^2)yx + x^2D(yx), \quad x, y \in R. \quad (3)$$

On the other hand the substitution $xy + yx$ for y in the relation (1) gives

$$G(x^2yx + xyx^2) = G(x)xyx + G(x)yx^2 + xD(xy x) + xD(yx^2), \quad x, y \in R. \quad (4)$$

From (3) and (4), we obtain

$$A(x)yx = 0 \quad x, y \in R, \quad (5)$$

where $A(x)$ stands for $G(x^2) - G(x)x - xD(x)$. We intend to prove that

$$A(x) = 0, \quad x \in R. \quad (6)$$

In relation (5) if we replace y by $xyA(x)$ then we get, $A(x)xyA(x)x = 0$, for all $x \in R$, and so, by the semiprimeness of R , we have

$$A(x)x = 0, \quad x \in R. \quad (7)$$

Multiplying the relation (5) from the left side by x and from the right by $A(x)$, we obtain $xA(x)yxA(x) = 0$ for all $x, y \in R$. By the semiprimeness of R it follows that

$$xA(x) = 0, \quad x \in R. \quad (8)$$

The linearization of the relation (7) gives

$$A(x)y + \beta(x, y)x + A(y)x + \beta(x, y)y = 0, \quad x, y \in R. \quad (9)$$

where $\beta(x, y)$ denotes $G(xy + yx) - G(x)y - G(y)x - xD(y) - yD(x)$. Putting in (9) $-x$ for x we get

$$A(x)y + \beta(x, y)x - A(y)x - \beta(x, y)y = 0, \quad x, y \in R. \quad (10)$$

Adding (9) to (10) and since R is a 2-torsion free we get

$$A(x)y + \beta(x, y)x = 0, \quad x, y \in R.$$

Right multiplication of the above equation by $A(x)$ gives $A(x)yA(x) = 0$, for all $x, y \in R$. Since R is semiprime, we get $A(x) = 0$, for all $x \in R$. We have therefore proved that $G(x^2) = G(x)x + xD(x)$ holds for all $x \in R$. In other words, G is a Jordan generalized derivation and the proof of the theorem is complete.

Now in view of Theorem 1.1 we obtain

Corollary 2.2 *Let R be a 2-torsion free semiprime ring such that R has a commutator which is not a zero divisor and let $G: R \longrightarrow R$ be an additive mapping. If $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$ and for a derivation D of R , then G is a generalized derivation.*

It is clear that if we used the derivation D to be the zero derivation on Theorem 2.1 we get

Corollary 2.3 *Let R be a 2-torsion free semiprime ring and let $T: R \longrightarrow R$ be an additive mapping. If $T(xy) = T(x)y$ for all $x, y \in R$, then T is a left centralizer.*

Now if R is a prime ring then we get the result of Molnár [4].

Corollary 2.4 *Let R be a 2-torsion free prime ring and let $T: R \longrightarrow R$ be an additive mapping. If $T(xy) = T(x)y$ for all $x, y \in R$, then T is a left centralizer.*

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