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On Generalized Derivations of Semiprime Rings with Involution

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Abstract

The purpose of this note is to prove the following result. Suppose R is a 2-torsion free semiprime *-ring such that R has a commutator which is not a zero divisor and let $G : R \to R$ be an additive mapping such that $G(xx^*) = G(x)x^* + xD(x^*)$ is fulfilled for all $x \in R$, for some derivation D of R. In this case G is a generalized derivation on R.

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1 Introduction

This note is inspired by the work of Molnár [6] and motivated by the work of Vukman and Kosi-Ulbl [7]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is 2-torsion free, if $2x = 0$, $x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $x \longrightarrow x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ holds for all $x, y \in R$. A ring equipped with an involution is called a ring with involution, or a $*$-ring. An additive mapping $D : R \longrightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [3] asserts that any Jordan derivation on 2-torsion free prime ring is a derivation. Cusack [2] generalized Herstein's theorem to 2-torsion free semiprime rings. An additive mapping $T : R \longrightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in R$. Zalar [8] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Vukman and Kosi-Ulbl [7] proved that if R is a 2-torsion free semiprime *-ring and $T : R \to R$ is an additive mapping such that $T(xx^*) = T(x)x^*$ $(T(x^*x) = x^*T(x))$ is fulfilled for all $x \in R$, then T is a left (right) centralizer.

In [4], Hvala has defined the notion of a generalized derivation as follows. An additive mapping $G : R \to R$ is said to be a generalized derivation if there exists a derivation $D : R \to R$ such that

$$G(xy) = G(x)y + xD(y) \text{ for all } x, y \in R.$$

Also, he called the map of the form $x \to ax + xb$ where a, b are fixed elements in R an inner generalized derivation. In [5] the authors have defined the notion of Jordan generalized derivations as follows. An additive mapping $G : R \to R$ is said to be a Jordan generalized derivation if there exists a derivation $D : R \to R$ such that

$$G(x^2) = G(x)x + xD(x) \text{ for all } x \in R.$$

Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in R$) and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizer (i.e., an additive map f satisfying $f(x^2) = f(x)x$ for all $x \in R$). In [1, Remark 1] Brešar proved that for a semiprime ring R, if G is a function from R to R and $D : R \to R$ is an additive mapping such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$, then D is uniquely determined by G and moreover G must be a derivation. Ashraf and Nadeem-Ur-Rehman, in [5], proved the following result:

Theorem 1.1 ([5], Theorem PP. 7). *Let R be a 2-torsion ring such that R has a commutator which is not a zero divisor. Then every Jordan generalized derivation on R is a generalized derivation.*

In this note we study a similar case to that of Vukman and Kosi-Ulbl [7] and will show that if R is a 2-torsion free semiprime *-ring such that R has a commutator which is not a zero divisor and G is an additive mapping that satisfies the identity,

$$G(xx^*) = G(x)x^* + xD(x^*)$$

for all $x \in R$ for some derivation D of R, then G is a generalized derivation on R.

In the sequel we will need the following result which are necessary for our proof,

Lemma 1.2 ([7]). *Let R be a semiprime *-ring. Suppose there exist an element $a \in R$, such that $ax^* = ax$ is fulfilled for all $x \in R$. In this case $a \in Z(R)$.*

2 Main Results

Our aim is to prove the following theorem

Theorem 2.1. *Let R be a 2-torsion free semiprime *-ring. Suppose there exists an additive mapping $G : R \to R$ related with some derivation D of R, such that*

$$G(xx^*) = G(x)x^* + xD(x^*) \tag{1}$$

is fulfilled for all $x \in R$. Then G is a Jordan generalized derivation.

Proof. Linearizing (1) gives

$$G(xy^* + yx^*) = G(x)y^* + G(y)x^* + yD(x^*) + xD(y^*), \quad x, y \in R. \tag{2}$$

Putting $y = x^*$ in (2) we get

$$\phi(x) + \phi(x^*) = 0, \quad x \in R, \tag{3}$$

where $\phi(x)$ stands for $G(x^2) - G(x)x - xD(x)$. Let $y = xy^* + yx^*$ in (2). We have

$$G(x(y + y^*)x^*) = -\phi(x)y^* + G(x)(y + y^*)x^* + xD((y + y^*)x^*), \quad x, y \in R. \tag{4}$$

Putting $y - y^*$ for y in (4) we obtain

$$\phi(x)y = \phi(x)y^*, \quad x, y \in R. \tag{5}$$

Relation (5) with Lemma 1.2 gives $\phi(x) \in Z(R)$ for all $x \in R$. Putting y^* for y in (2) we obtain

$$G(xy + y^*x^*) = G(x)y + G(y^*)x^* + y^*D(x^*) + xD(y), \quad x, y \in R. \tag{6}$$

The substitution xy for y in (6) gives

$$G(x^2y + y^*x^{*2}) = G(x)xy + G(y^*x^*)x^* + y^*x^*D(x^*) + xD(xy), \quad x, y \in R. \tag{7}$$

On the other hand the substitution x^2 for x in (6) leads to

$$G(x^2y + y^*x^{*2}) = G(x^2)y + G(y^*)x^{*2} + y^*D(x^{*2}) + x^2D(y), \quad x, y \in R. \tag{8}$$

Subtracting (7) from (8) we get

$$\phi(x)y + (G(y^*)x^* - G(y^*x^*) + y^*D(x^*))x^* = 0, \quad x, y \in R. \tag{9}$$

In particular for $y = x$ the relation (9) reduced to

$$\phi(x)x - \phi(x^*)x^* = 0, \quad x \in R. \tag{10}$$

According to (3) we can obtain

$$\phi(x)(x + x^*) = 0, \quad x \in R. \tag{11}$$

Putting $y = x$ in (5) we obtain

$$\phi(x)(x - x^*) = 0, \quad x \in R. \tag{12}$$

Combining the equations (11) and (12) we have

$$\phi(x)x = 0, \quad x \in R. \tag{13}$$

Since $\phi(x) \in Z(R)$, $x \in R$, we also have

$$x\phi(x) = 0, \quad x \in R. \tag{14}$$

Linearizing of (13) gives

$$\phi(x)y + \phi(y)x + \psi(x, y)x + \psi(x, y)y = 0, \quad x, y \in R, \tag{15}$$

where $\psi(x, y)$ stands for $G(xy+yx)-G(x)y-G(y)x-xD(y)-yD(x)$. Putting in the above equation $-x$ for x and comparing the relation so obtained with the above equation we obtain, since R is 2-torsion free,

$$\phi(x)y + \psi(x, y)x = 0, \quad x, y \in R.$$

Right multiplication of the above relation by $\phi(x)$ gives $\phi(x)y\phi(x) = 0$, $x, y \in R$, because of (14). Whence it follows $\phi(x) = 0$. In other words, G is a Jordan generalized derivation and our proof is complete.

In view of Theorem 1.1 we can get

Corollary 2.2. *Let R be a 2-torsion free semiprime *-ring such that R has a commutator which is not a zero divisor. Suppose there exist an additive mapping $G : R \rightarrow R$ related with some derivation D of R, such that*

$$G(xx^*) = G(x)x^* + xD(x^*)$$

is fulfilled for all $x \in R$. Then G is a generalized derivation.

It is clear that if we take the derivation D to be the zero derivation, in Theorem 2.1, we get

Corollary 2.3 ([7], Theorem 1). *Let R be a 2-torsion free semiprime *-ring and let $T\colon R \longrightarrow R$ be an additive mapping. If $T(xx^*) = T(x)x^*$ for all $x \in R$, then T is a left (right) centralizer.*

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