On Generalized Derivations in Semiprime Rings

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Abstract

The purpose of this note is to prove the following result. Let $R$ be a semiprime ring of characteristic not 2 and $G: R \rightarrow R$ be an additive mapping such that $G(x^2) = G(x)x + xD(x)$ holds for all $x \in R$ and some derivations $D$ of $R$. Then $G$ is a Jordan generalized derivation.

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1 Introduction

This note is motivated by the work of Zalar [6]. Throughout, $R$ will represent an associative ring with center $Z(R)$. A ring $R$ is $n$-torsion free, if $nx = 0$, $x \in R$ implies $x = 0$, where $n$ is a positive integer. Recall that $R$ is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $T: R \rightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$ and is called a Jordan left (right) centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) holds for all $x \in R$. A result of Zalar [6] asserts that any Jordan centralizer on a semiprime ring of characteristic not 2 is a centralizer. An additive mapping $D: R \rightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$.
and is called a Jordan derivation in case \( D(x^2) = D(x)x + xD(x) \) holds for all \( x \in R \). A derivation \( D \) is inner if there exists \( a \in R \) such that \( D(x) = ax - xa \) holds for all \( x \in R \). Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [4] asserts that any Jordan derivation on 2-torsion free prime ring is a derivation. Cusack [2] generalized Herstein’s theorem to 2-torsion free semiprime ring.

In [3], Hvala has defined the notion of a generalized derivation as follows: An additive mapping \( G : R \to R \) is said to be a generalized derivation if there exists a derivation \( D : R \to R \) such that \( G(xy) = G(x)y + xD(y) \) for all \( x, y \in R \). Also, he called the maps of the form \( x \to ax + xb \) where \( a, b \) are fixed elements in \( R \) by the inner generalized derivations. Ashraf and Nadeem-Ur-Rehman, in [5], have defined the concept of a Jordan generalized derivation as follows: An additive mapping \( G : R \to R \) is said to be a Jordan generalized derivation if there exists a derivation \( D : R \to R \) such that \( G(x^2) = G(x)x + xD(x) \) for all \( x \in R \). Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizers and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizers. In [1, Remark 1] Brešar proved that for a semiprime ring \( R \), if \( G \) is a function from \( R \) to \( R \) and \( D : R \to R \) is an additive mapping such that \( G(xy) = G(x)y + xD(y) \) for all \( x, y \in R \), then \( D \) is uniquely determined by \( G \) and moreover \( G \) must be a derivation. Ashraf and Nadeem-Ur-Rehman, in [5], proved the following result: Let \( R \) be a 2-torsion free ring such that \( R \) has a commutator which is not a zero divisor, then every Jordan generalized derivation on \( R \) is a generalized derivation.

In this note, using Zalar’s method, we study the same result of Ashraf and Nadeem-Ur-Rehman but for a semiprime ring, and without the condition of zero divisor, i.e., if \( R \) is a semiprime ring of characteristic not 2 and \( G \) is an additive mapping which satisfies

\[
G(x^2) = G(x)x + xD(x)
\]

holds for all \( x \in R \) and some derivation \( D \) of \( R \), then \( G \) is a Jordan generalized derivation. This result will be a generalization of the result of Zalar [6]. In order to prove our result we will need the following lemmas which are due to Zalar.

**Lemma 1.1 ([6] Lemma 1.1).** Let \( R \) be a semiprime ring. If \( a, b \in R \) are such that \( axb = 0 \) for all \( x \in R \), then \( ab = ba = 0 \).

**Lemma 1.2 ([6] Lemma 1.2).** Let \( R \) be a semiprime ring and \( \theta, \phi : R \times R \to R \) biadditive mappings. If \( \theta(x, y)w\phi(x, y) = 0 \) for all \( x, y, w \in R \), then \( \theta(x, y)w\phi(u, v) = 0 \) for all \( x, y, u, v, w \in R \).
Lemma 1.3 ([6] Lemma 1.3). Let $R$ be a semiprime ring and $a \in R$ be some fixed element. If $a[x, y] = 0$ for all $x, y \in R$, then there exists an ideal $U$ of $R$ such that $a \in U \subset Z(R)$ holds.

2 The Main Result

Theorem 2.1. Let $R$ be a semiprime ring of characteristic not 2 and $G: R \rightarrow R$ be an additive mapping satisfying the relation

\[ G(x^2) = G(x)x + xD(x), \]

for all $x \in R$ and some derivation $D$ of $R$. Then $G$ is a Jordan generalized derivation.

Proof. Replacing $x$ by $x + y$ in (1) we get

\[ G(xy + yx) = G(x)y + G(y)x + xD(y) + yD(x), \quad x, y \in R. \] (2)

Replacing $y$ by $xy + yx$ in (2) and using (2) we obtain

\[ G(x^2y + yx^2) + 2G(xy) = G(x)xy + G(x)yx + G(x)y^2 + xD(y)x + yD(x)x + xD(xy + yx) + (xy + yx)D(x), \quad x, y \in R. \] (3)

On the other hand, replacing $x$ by $x^2$ in (2) and adding $2G(xy)$ to both sides we get

\[ G(x^2y + yx^2) + 2G(xy) = G(x)xy + xD(x)y + G(y)x^2 + x^2D(y) + yxD(x) + yD(x)x + 2G(xy), \quad x, y \in R. \] (4)

Comparing (3) and (4) we obtain

\[ G(xy) = G(x)yx + xD(yx), \quad x, y \in R. \] (5)

Putting $x = x + z$ in (5), we get

\[ G(xyz + zyx) = G(x)yz + G(z)yx + xD(yz) + zD(yx), \quad x, y, z \in R. \] (6)

Let $F = G(xyz + yxz)$, we shall compute it in two different ways. Using (5) we have

\[ F = G(x)xyz + G(y)xzxy + xD(yzx) + yD(xzxy), \quad x, y, z \in R. \] (7)
Using (6) we have
\[ F = G(xy)zx + G(yx)zy + xyD(zyx) + yxD(zxy), \quad x, y, z \in R. \] (8)
Comparing (7) and (8) we get
\[ \theta(x, y)zx + \theta(y, x)zy = 0, \quad x, y, z \in R, \] (9)
where \( \theta(x, y) \) stands for \( G(xy) - G(x)y - xD(y) \). In the concept of the definition of \( \theta \), equation (2) can be rewritten in the form \( \theta(x, y) = -\theta(y, x) \). Using this notation in equation (9) we get
\[ \theta(x, y)z[x, y] = 0, \quad x, y, z \in R. \] (10)
Using Lemma 1.2 we get
\[ \theta(x, y)z[u, v] = 0, \quad x, y, z, u, v \in R. \] (11)
Using Lemma 1.1 we obtain
\[ \theta(x, y)[u, v] = 0, \quad x, y, u, v \in R. \] (12)
Now fix \( x, y \in R \) and write \( \theta \) instead of \( \theta(x, y) \) to simplify further writing. Using Lemma 1.3 we get the existence of an ideal \( U \) such that \( \theta \in U \subset Z(R) \) holds. In particular, \( b\theta, \theta b \in Z(R) \) for all \( b \in R \). This gives us
\[ x\theta^2 y = \theta^2 y.x = y \theta^2 .x = y, \theta^2 x. \]
This gives us \( 4G(x, \theta^2 y) = 4G(y, \theta^2 x). \) Now we will compute each side of this equality by using (2) and the above notation.
\[ 4G(x, \theta^2 y) = 2G(x\theta^2 y + \theta^2 y) = \]
\[ = 2G(x)\theta^2 y + 2xD(\theta^2 y) + 2G(\theta^2 y)x + 2\theta^2 yD(x) = \]
\[ = 2G(x)\theta^2 y + G(\theta^2 y + \theta^2) + 2xD(\theta^2 y)x + 2\theta^2 yD(x) = \]
\[ = 2G(x)\theta^2 y + G(\theta)\theta y x + \theta D(\theta)y x + G(y)\theta^2 x + \theta^2 D(y)x + yD(\theta^2)x + 2xD(\theta^2 y) + 2\theta^2 yD(x). \]
So we get
\[ 4G(x, \theta^2 y) = 2G(x)\theta^2 y + G(\theta)\theta y x + \theta D(\theta)y x + G(y)\theta^2 x + \theta^2 D(y)x + yD(\theta^2)x + 2xD(\theta^2 y) + 2\theta^2 yD(x), \quad x, y \in R. \] (13)
Moreover,
4G(y.θ^2x) = 2G(yθ^2x + θ^2xy) =
= 2G(y)θ^2x + 2yD(θ^2x) + 2G(θ^2x)y + 2θ^2xD(y) =
= 2G(y)θ^2x + G(θ^2x + xθ^2)y + 2yD(θ^2x) + 2θ^2xD(y) =
= 2G(y)θ^2x + G(θ)θxy + θD(θ)xy + G(x)θ^2y + θ^2D(x)y + xD(θ^2)y +
2yD(θ^2x) + 2θ^2xD(y).

So we get

4G(y.θ^2x) = 2G(y)θ^2x + G(θ)θxy + θD(θ)xy + G(x)θ^2y + θ^2D(x)y +
xD(θ^2)y + 2yD(θ^2x) + 2θ^2xD(y), \quad x, y \in R. \quad (14)

Comparing (13) and (14) and using the following notations

θyx = θy.x = xθy = xy,
θD(θ)yx = D(θ)θyx = D(θ)θy = θD(θ)xy,
xD(θ)y = D(θ)xθy = D(θ)θxy = D(θ)θy = θyD(θ)x = yθD(θ)x,

we obtain

G(x)θ^2y + xθ^2D(y) = G(y)θ^2x + yθ^2D(x)

which gives

φ(x, y)θ^2 = φ(y, x)θ^2, \quad (15)

where φ(x, y) stands for G(x)y + xD(y). On the other hand, we also have
4G(xyθ^2) = 4G(xθ.yθ). We will compute each side of this equality by using
(2) and the properties of θ, so we get

4G(xyθ^2) = 2G(xyθ^2 + θ^2xy) = 2G(xy)θ^2 + xD(θ^2) + 2G(θ^2)xy + 2θ^2D(xy),

which gives

4G(xyθ^2) = 2G(xy)θ^2 + 2xyD(θ^2) + 2G(θ^2)xy + 2θ^2D(xy), \quad x, y \in R. \quad (16)

Moreover,

4G(xθ.yθ) = 2G(xθyθ + yθxθ) =
= G(xθ + θx)y + 2θxD(θy) + G(yθ + θy)θx + 2θyD(θx) =
\[ G(x)\theta^2 y + G(\theta)\theta xy + xD(\theta)\theta y + \theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2 x + G(\theta)\theta y x + yD(\theta)\theta x + \theta D(y)\theta x + 2\theta yD(\theta x). \]

So we obtain

\[ 4G(x\theta, y\theta) = G(x)\theta^2 y + G(\theta)\theta xy + xD(\theta)\theta y + \theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2 x + G(\theta)\theta y x + yD(\theta)\theta x + \theta D(y)\theta x + 2\theta yD(\theta x), \quad x, y \in R. \]  

(17)

Comparing (16) and (17), we obtain

\[ 2G(xy)\theta^2 = \phi(x, y)\theta^2 + \phi(y, x)\theta^2, \quad x, y \in R. \]  

(18)

Using (15), finally we get \( G(xy)\theta^2 = \phi(x, y)\theta^2 \). But \( \theta(x, y) = G(xy) - \phi(x, y) \) and this means \( \theta^3 = 0 \) so that

\[ \theta^2 R\theta^2 = \theta^4 R = (0), \]
\[ \theta R\theta = \theta^2 R = (0), \]

which implies \( \theta = 0 \), and the proof is complete.

It is clear that if we let the derivation \( D \) to be the zero derivation in the above theorem, we get the following result.

**Corollary 2.2 ([6] Proposition 1.4).** Let \( R \) be a semiprime ring of characteristic not 2 and \( T : R \to R \) an additive mapping which satisfies \( T(x^2) = T(x)x \) for all \( x \in R \). Then \( T \) is a left centralizer.

**References**


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