An identity related to centralizers in semiprime rings

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Abstract. The purpose of this paper is to prove the following result: Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping, such that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$. In this case T is left and right centralizer.

Keywords: prime ring, semiprime ring, derivation, Jordan derivation, left (right) centralizer, left (right) Jordan centralizer

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This research has been motivated by the work of Brešar [2] and Zalar [8]. Throughout, R will represent an associative ring with the center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$ and use basic commutator identity $[xy, z] = [x, z]y + x[y, z]$, $x, y, z \in R$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. A classical result of Herstein [4] states that in case R is a prime ring of characteristic different from two, then every Jordan derivation is a derivation. A brief proof of Herstein's result can be found in [1]. Cusak [3] has generalized Herstein's result on 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y \ \ (T(xy) = xT(y))$ holds for all pairs $x, y \in R$.

In case R has an identity element $T : R \to R$ is a left (right) centralizer iff T is of the form $T(x) = ax \ \ (T(x) = xa)$ for some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x \ \ (T(x^2) = xT(x))$ is fulfilled for all $x \in R$. Following ideas from [2], Zalar has proved in [8] that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Some results concerning centralizers in prime and semiprime rings can be found in our recent paper [7].

Let us start with the following result proved by Zalar in [8].

Theorem A ([8, Proposition 1.4]). *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be a left (right) Jordan centralizer. In this case T is left (right) centralizer.*

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If the mapping $T : R \to R$, where R is an arbitrary ring, is both left and right Jordan centralizer, then obviously T satisfies the relation

$$2T(x^2) = T(x)x + xT(x), \quad x \in R.$$

It seems natural to ask whether additive mapping which satisfies the relation above is left and right Jordan centralizer. This question leads to the following result.

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be such an additive mapping that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$. In this case T is left and right centralizer.*

Let us point out that in case R has an identity element, Theorem A can be easily proved for an arbitrary ring. Namely, in this case one puts $x + 1$ for x in $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$, where 1 denotes the identity element, which gives $T(x) = T(1)x$ $(T(x) = xT(1))$. Thus T is left (right) centralizer. However, in case of Theorem 1 it seems that the proof is nontrivial even in case we have a semiprime ring with an identity element. We intend to present the proof of Theorem 1 for the case R has an identity element since it might be of some independent interest and since it is, as we shall see, related to some classical results in prime and semiprime ring theory.

Proof of Theorem 1 in case R has an identity element.

We shall assume that R is noncommutative, since there is not much to prove in case R is commutative. Putting in the relation

$$2T(x^2) = T(x)x + xT(x), \quad x \in R, \tag{1}$$

$x + 1$ for x we obtain after some calculations

$$2T(x) = ax + xa, \quad x \in R, \tag{2}$$

where a denotes $T(1)$. We intend to prove that $a \in Z(R)$. Combining (1) and (2) we obtain $2(ax^2 + x^2a) = (ax + xa)x + x(ax + xa)$, which reduces to

$$[D(x), x] = 0, \quad x \in R, \tag{3}$$

where $D(x)$ stands for $[a, x]$. Here we meet the theory of so-called centralizing and commuting mappings. A mapping F of a ring R into itself is called centralizing on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[F(x), x] = 0$ holds for all $x \in R$, the mapping F is said to be commuting on R. The study of centralizing and commuting mappings was initiated by the classical result of Posner [6], which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). Therefore, in case R is a prime ring, it follows from (3) and Posner's second theorem that $D = 0$, since we have assumed that R is noncommutative. In other words in this case we have $a \in Z(R)$. But we have the weaker assumption that R is a semiprime ring and in semiprime rings Posner's second theorem in general does not hold as the following simple example shows. Take R_1 to be a noncommutative prime ring and R_2 to be a commutative prime ring that admits a nonzero derivation $d : R_2 \to R_2$. Then $R_1 \times R_2$ is a noncommutative semiprime ring and the mapping $D : R_1 \times R_2 \to R_1 \times R_2$, $D(r_1, r_2) = (0, d(r_2))$ is a nonzero derivation which maps $R_1 \times R_2$ into its center. In the sequel we shall prove that in case we have an inner derivation D, which is commuting on a noncommutative semiprime ring, then $D = 0$. For this purpose we put $x + y$ for x in (3). We have $[D(x), y] + [D(y), x] = 0$, $x, y \in R$, and in particular for $y = a$, since $D(a) = 0$, we obtain $[D(x), a] = 0$, $x \in R$. In other words we have

$$D^2(x) = 0, \quad x \in R. \tag{4}$$

Posner's first theorem [6] asserts that if R is a prime ring of characteristic different from two and D, G are nonzero derivations on R, then DG cannot be a derivation. Let us point out that Posner's first theorem does not hold for semiprime rings. However, it is well known that if D and D^2 are derivations in 2-torsion free semiprime ring, then $D = 0$ (see the proof of Lemma 1.1.9 in [5]). Hence we have $D = 0$. In other words $a \in Z(R)$. Now it follows from (2) that $T(x) = ax$, and $T(x) = xa$ for all $x \in R$, which completes the proof of the theorem in case R has an identity element.

In the sequel we present the proof of Theorem 1 without assuming that R has an identity element. The proof is, as we shall see, rather long but it is elementary in the sense that it requires no specific knowledge concerning semiprime rings in order to follow the proof.

Proof of Theorem 1 – general case.

We intend to prove that T is commuting on R. In other words, it is our aim to prove that

$$[T(x), x] = 0, \tag{5}$$

holds for all $x \in R$. In order to achieve this goal we shall first prove a weaker result that T satisfies the relation

$$[T(x), x^2] = 0, \quad x \in R. \tag{6}$$

Since the above relation can be written in the form $[T(x), x]x + x[T(x), x] = 0$, it is obvious that T satisfies the relation (6) in case T is commuting on R.

Putting in the relation (1) $x + y$ for x one obtains

$$2T(xy + yx) = T(x)y + xT(y) + T(y)x + yT(x), \quad x, y \in R. \tag{7}$$

Our next step is to prove the relation

$$(8)\quad 8T(xyx) = T(x)(xy+3yx) + (yx+3xy)T(x) + 2xT(y)x - x^2T(y) - T(y)x^2, \quad x,y \in R.$$

For this purpose we put in the relation (7) $2(xy+yx)$ for y. Then using (7) we obtain

$$\begin{aligned} 4T(x(xy+yx)+(xy+yx)x) &= 2T(x)(xy+yx) + 2xT(xy+yx) + 2T(xy+yx)x \\ &+ 2(xy+yx)T(x) = 2T(x)(xy+yx) + xT(x)y + x^2T(y) + xT(y)x \\ &+ xyT(x) + T(x)yx + xT(y)x + T(y)x^2 + yT(x)x + 2(xy+yx)T(x). \end{aligned}$$

Thus we have

$$(9)\quad 4T(x(xy+yx)+(xy+yx)x) = T(x)(2xy+3yx) + (3xy+2yx)T(x) + xT(x)y + yT(x)x + 2xT(y)x + x^2T(y) + T(y)x^2, \quad x,y \in R.$$

On the other hand, using (7) and (1) we obtain

$$\begin{aligned} 4T(x(xy+yx)+(xy+yx)x) &= 4T(x^2y+yx^2) + 8T(xyx) = 2T(x^2)y \\ &+ 2x^2T(y) + 2T(y)x^2 + 2yT(x^2) + 8T(xyx) = T(x)xy + xT(x)y + 2x^2T(y) \\ &+ 2T(y)x^2 + yT(x)x + yxT(x) + 8T(xyx). \end{aligned}$$

We have therefore

$$(10)\quad 4T(x(xy+yx)+(xy+yx)x) = T(x)xy + yxT(x) + xT(x)y + yT(x)x + 2x^2T(y) + 2T(y)x^2 + 8T(xyx), \quad x,y \in R.$$

By comparing (9) with (10) we arrive at (8). Let us prove the relation

$$(11)\quad T(x)(xyx - 2yx^2 - 2x^2y) + (xyx - 2x^2y - 2yx^2)T(x) + xT(x)(xy+yx) + (xy+yx)T(x)x + x^2T(x)y + yT(x)x^2 = 0, \quad x,y \in R.$$

Putting in (7) $8xyx$ for y and using (8) we obtain

$$\begin{aligned} 16T(x^2yx + xyx^2) &= 8T(x)xyx + 8xT(xyx) + 8T(xyx)x + 8xyxT(x) \\ &= 8T(x)xyx + xT(x)(xy+3yx) + (xyx+3x^2y)T(x) + 2x^2T(y)x - x^3T(y) \\ &- xT(y)x^2 + T(x)(xyx+3yx^2) + (yx+3xy)T(x)x + 2xT(y)x^2 - x^2T(y)x \\ &- T(y)x^3 + 8xyxT(x). \end{aligned}$$

We have therefore

$$(12)\quad 16T(x^2yx+xyx^2)=T(x)(9xyx+3yx^2)+(9xyx+3x^2y)T(x)$$
$$+xT(x)(xy+3yx)+(yx+3xy)T(x)x+x^2T(y)x+xT(y)x^2$$
$$-T(y)x^3-x^3T(y),\quad x,y\in R.$$

On the other hand, we obtain first using (8) and then after collecting some terms using (7)

$$16T(x^2yx+xyx^2)=16T(x(xy)x)+16T(x(yx)x)$$
$$=2T(x)(x^2y+3xyx)+2(xyx+3x^2y)T(x)+4xT(xy)x-2x^2T(xy)$$
$$-2T(xy)x^2+2T(x)(xyx+3yx^2)+2(yx^2+3xyx)T(x)+4xT(yx)x$$
$$-2x^2T(yx)-2T(yx)x^2=T(x)(2x^2y+6yx^2+8xyx)+(8xyx+2yx^2$$
$$+6x^2y)T(x)+4xT(xy+yx)x-2x^2T(xy+yx)-2T(xy+yx)x^2$$
$$=T(x)(2x^2y+6yx^2+8xyx)+(8xyx+2yx^2+6x^2y)T(x)+2xT(x)yx$$
$$+2x^2T(y)x+2xT(y)x^2+2xyT(x)x-x^2T(x)y-x^3T(y)$$
$$-x^2T(y)x-x^2yT(x)-T(x)yx^2-xT(y)x^2-T(y)x^3-yT(x)x^2.$$

We have therefore

$$(13)\quad 16T(x^2yx+xyx^2)=T(x)(2x^2y+5yx^2+8xyx)$$
$$+(2yx^2+5x^2y+8xyx)T(x)+2xT(x)yx+2xyT(x)x+x^2T(y)x+xT(y)x^2$$
$$-x^2T(x)y-yT(x)x^2-x^3T(y)-T(y)x^3,\quad x,y\in R.$$

By comparing (12) with (13) we obtain (11).

Replacing in (11) y by yx we obtain

$$(14)\quad T(x)(xyx^2-2yx^3-2x^2yx)+(xyx^2-2x^2yx-2yx^3)T(x)$$
$$+xT(x)(xyx+yx^2)+(xyx+yx^2)T(x)x+x^2T(x)yx+yxT(x)x^2=0,$$
$$x,y\in R.$$

Right multiplication of (11) by x gives

$$(15)\quad T(x)(xyx^2-2yx^3-2x^2yx)+(xyx-2x^2y-2yx^2)T(x)x$$
$$+xT(x)(xyx+yx^2)+(xy+yx)T(x)x^2+x^2T(x)yx+yT(x)x^3=0,$$
$$x,y\in R.$$

Subtracting (15) from (14) we obtain $xyx[x,T(x)]+2x^2y[T(x),x]+2yx^2[T(x),x]+xy[x,T(x)]+yx[x,T(x)]x+y[x,T(x)]x^2=0$, $x,y\in R$ which reduces after collecting the first and the fourth term together to

$$(16)\quad xy[x^2,T(x)]+2x^2y[T(x),x]+2yx^2[T(x),x]+yx[x,T(x)]$$
$$+y[x,T(x)]x^2=0,\quad x,y\in R.$$

Substituting $T(x)y$ for y in the above relation gives

$$(17)\quad xT(x)y[x^2,T(x)] + 2x^2T(x)y[T(x),x] + 2T(x)yx^2[T(x),x] + T(x)yx[x,T(x)]x + T(x)y[x,T(x)]x^2 = 0, \quad x,y \in R.$$

Left multiplication of (16) by $T(x)$ leads to

$$(18)\quad T(x)xy[x^2,T(x)] + 2T(x)x^2y[T(x),x] + 2T(x)yx^2[T(x),x] + T(x)yx[x,T(x)]x + T(x)y[x,T(x)]x^2 = 0, \quad x,y \in R.$$

Subtracting (18) from (17) we arrive at

$$[T(x),x]y[T(x),x^2] - 2[T(x),x^2]y[T(x),x] = 0, \quad x,y \in R.$$

We set

$$a = [T(x),x], \quad b = [T(x),x^2], \quad c = -2[T(x),x^2].$$

Then the above relation becomes

$$(19)\qquad ayb + cya = 0, \quad y \in R.$$

Putting in (19) yaz for y we obtain

$$(20)\qquad ayazb + cyaza = 0, \quad z,y \in R.$$

Left multiplication of (19) by ay gives

$$(21)\qquad ayazb + aycza = 0, \quad z,y \in R.$$

Subtracting (21) from (20) we obtain

$$(22)\qquad (ayc - cya)za = 0, \quad z,y \in R.$$

Let in (22) z be zcy we obtain

$$(23)\qquad (ayc - cya)zcya = 0, \quad z,y \in R.$$

Right multiplication of (22) by yc gives

$$(24)\qquad (ayc - cya)zayc = 0, \quad z,y \in R.$$

Subtracting (23) from (24) we obtain $(ayc - cya)z(ayc - cya) = 0$, $z,y \in R$, whence it follows

$$(25)\qquad ayc = cya, \quad y \in R.$$

Combining (19) with (25) we arrive at

$$ay(b+c)=0, \quad y \in R.$$

In other words

$$[T(x),x]y[T(x),x^2]=0, \quad x,y \in R. \tag{26}$$

From the above relation one obtains easily $([T(x),x]x+x[T(x),x])y[T(x),x^2]=0$, $x,y \in R$. We have therefore

$$[T(x),x^2]y[T(x),x^2]=0, \quad x,y \in R,$$

which implies

$$[T(x),x^2]=0, \quad x \in R. \tag{27}$$

Substitution $x+y$ for x in (27) gives

$$[T(x),y^2]+[T(y),x^2]+[T(x),xy+yx]+[T(y),xy+yx]=0, \quad x,y \in R.$$

Putting in the above relation $-x$ for x and comparing the relation so obtained with the above relation we obtain, since we have assumed that R is 2-torsion free

$$[T(x),xy+yx]+[T(y),x^2]=0, \quad x,y \in R. \tag{28}$$

Putting in the above relation $2(xy+yx)$ for y we obtain according to (7) and (27)

$$\begin{aligned}0&=2[T(x),x^2y+yx^2+2xyx]+[T(x)y+xT(y)+T(y)x+yT(x),x^2]\\&=2x^2[T(x),y]+2[T(x),y]x^2+4[T(x),xyx]+T(x)[y,x^2]+x[T(y),x^2]\\&\qquad+[T(y),x^2]x+[y,x^2]T(x).\end{aligned}$$

Thus we have

$$\begin{aligned}&2x^2[T(x),y]+2[T(x),y]x^2+4[T(x),xyx]+T(x)[y,x^2]+[y,x^2]T(x)\\&\qquad+x[T(y),x^2]+[T(y),x^2]x=o, \quad x,y \in R.\end{aligned} \tag{29}$$

For $y=x$ the above relation reduces to

$$x^2[T(x),x]+[T(x),x]x^2+2[T(x),xx^2]=0,$$

which gives

$$x^2[T(x),x]+3[T(x),x]x^2=0, \quad x \in R.$$

According to the relation $[T(x), x]x + x[T(x), x] = 0$ (see (27)) one can replace in the above relation $x^2[T(x), x]$ by $[T(x), x]x^2$, which gives

$$[T(x), x]x^2 = 0, \quad x \in R \tag{30}$$

and

$$x^2[T(x), x] = 0, \quad x \in R. \tag{31}$$

We have also

$$x[T(x), x]x = 0, \quad x \in R. \tag{32}$$

Because of (28) one can replace in (29) $[T(y), x^2]$ by $-[T(x), xy + yx]$, which gives

$$\begin{aligned} 0 = 2x^2[T(x), y] &+ 2[T(x), y]x^2 + 4[T(x), xyx] + T(x)[y, x^2] + [y, x^2]T(x) \\ &- x[T(x), xy + yx] - [T(x), xy + yx]x = 2x^2[T(x), y] + 2[T(x), y]x^2 \\ &+ 4[T(x), x]yx + 4x[T(x), y]x + 4xy[T(x), x] + T(x)[y, x^2] + [y, x^2]T(x) \\ &- x[T(x), x]y - x^2[T(x), y] - x[T(x), y]x - xy[T(x), x] - [T(x), x]yx \\ &- x[T(x), y]x - [T(x), y]x^2 - y[T(x), x]x. \end{aligned}$$

We have therefore

$$\begin{aligned} &x^2[T(x), y] + [T(x), y]x^2 + 3[T(x), x]yx + 3xy[T(x), x] + 2x[T(x), y]x \\ &\quad + T(x)[y, x^2] + [y, x^2]T(x) + x[T(x), x]y - y[T(x), x]x = 0, \quad x, y \in R. \end{aligned} \tag{33}$$

The substitution yx for y gives

$$\begin{aligned} 0 &= x^2[T(x), yx] + [T(x), yx]x^2 + 3[T(x), x]yx^2 + 3xyx[T(x), x] \\ &\quad + 2x[T(x), yx]x + T(x)[yx, x^2] + [yx, x^2]T(x) + x[T(x), x]yx - yx[T(x), x]x \\ &= x^2[T(x), y]x + x^2y[T(x), x] + [T(x), y]x^3 + y[T(x), x]x^2 + 3[T(x), x]yx^2 \\ &\quad + 3xyx[T(x), x] + 2x[T(x), y]x^2 + 2xy[T(x), x]x + T(x)[y, x^2]x \\ &\quad + [y, x^2]xT(x) + x[T(x), x]yx - yx[T(x), x]x, \end{aligned}$$

which reduces because of (30) and (32) to

$$\begin{aligned} &x^2[T(x), y]x + x^2y[T(x), x] + [T(x), y]x^3 + 3[T(x), x]yx^2 \\ &\quad + 3xyx[T(x), x] + 2x[T(x), y]x^2 + 2xy[T(x), x]x + T(x)[y, x^2]x \\ &\quad + [y, x^2]xT(x) + x[T(x), x]yx = 0, \quad x, y \in R. \end{aligned} \tag{34}$$

Right multiplication of (33) by x gives

$$x^2[T(x),y]x+[T(x),y]x^3+3[T(x),x]yx^2+3xy[T(x),x]x+2x[T(x),y]x^2 + T(x)[y,x^2]x+[y,x^2]T(x)x+x[T(x),x]yx=0, \quad x,y\in R. \tag{35}$$

Subtracting (35) from (34) we obtain

$$x^2y[T(x),x]+3xy[x,[T(x),x]]+2xy[T(x),x]x+[y,x^2][x,T(x)]=0,$$

which reduces because of (31) to

$$2x^2y[T(x),x]+3xyx[T(x),x]-xy[T(x),x]x=0, \quad x,y\in R.$$

Replacing in the above relation $-[T(x),x]x$ by $x[T(x),x]$ we obtain

$$x^2y[T(x),x]+2xyx[T(x),x]=0, \quad x,y\in R.$$

Because of (27), (30), (31) and (32) the relation (16) reduces to $x^2y[T(x),x]=0$, $x,y\in R$, which gives together with the relation above $xyx[T(x),x]=0$, $x,y\in R$, whence it follows

$$x[T(x),x]yx[T(x),x]=0, \quad x,y\in R.$$

Thus we have

$$x[T(x),x]=0, \quad x\in R. \tag{36}$$

Of course we have also

$$[T(x),x]x=0, \quad x\in R. \tag{37}$$

From (36) one obtains (see the proof of (28))

$$y[T(x),x]+x[T(x),y]+x[T(y),x]=0, \quad x,y\in R.$$

Left multiplication of the above relation by $[T(x),x]$ gives because of (37)

$$[T(x),x]y[T(x),x]=0, \quad x,y\in R,$$

whence it follows

$$[T(x),x]=0, \quad x\in R. \tag{38}$$

Combining (38) with (1) we obtain

$$T(x^2)=T(x)x, \quad x\in R,$$

and also

$$T(x^2)=xT(x), \quad x\in R,$$

which means that T is left and also right Jordan centralizer. By Proposition 1.4 in [8] T is both left and also right centralizer. The proof of the theorem is complete. □

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