Larson–Sweedler Theorem, Grouplike Elements, Invertible Modules and the Order of the Antipode in Weak Hopf Algebras

Peter Vecsernyés

Research Institute for Particle and Nuclear Physics, Budapest

H-1525 Budapest 114, P.O.B. 49, Hungary

Abstract

We extend the Larson–Sweedler theorem for weak Hopf algebras by proving that a finite dimensional weak bialgebra is a weak Hopf algebra iff it possesses a non-degenerate left integral. We establish the autonomous monoidal category of the modules of a weak Hopf algebra A and show the semisimplicity of the unit and the invertible modules of A. We also reveal the connection of these modules to left/right grouplike elements in the dual weak Hopf algebra $\hat{A}$. Defining distinguished left/right grouplike elements we derive the Radford formula for the fourth power of the antipode in a weak Hopf algebra and prove that the order of the antipode is finite up to an inner automorphism by a grouplike element in the trivial subalgebra A^T of the underlying weak Hopf algebra A.

E-mail: vecser@rmki.kfki.hu

Supported by the Hungarian Research Fund, OTKA – T 034 512

0. Introduction

Weak Hopf algebras have been proposed recently [1, 2, 18] as a generalization of Hopf algebras by weakening the compatibility conditions between the algebra and coalgebra structures of Hopf algebras.

Similarly to weak quasi Hopf algebras [11] and rational Hopf algebras [19, 8] the comultiplication is allowed to be non-unital, $\Delta(\mathbf{1}) \equiv \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \neq \mathbf{1} \otimes \mathbf{1}$, but in contrast the comultiplication is coassociative. In exchange for coassociativity the multiplicativity of the counit is replaced by a weaker condition: $\varepsilon(ab) = \varepsilon(a\mathbf{1}^{(1)})\varepsilon(\mathbf{1}^{(2)}b)$, implying that the unit representation is not necessarily one-dimensional and irreducible. Similarly to weak quasi and rational Hopf algebras they can possess non-integral (quantum) dimensions even in the finite dimensional and semisimple cases, which is necessary if we would like to recover them as global symmetries of low-dimensional quantum field theories. In situations where only the representation category matters these two concepts are equivalent. Nevertheless, finite dimensional weak Hopf algebras (WHA) obey the mathematical beauty that, similarly to finite dimensional Hopf algebras, they give rise to a self-dual notion: the dual space of a WHA can be canonically endowed with a WHA structure. For a recent review, see [12].

Here we continue the study [2] of the structural properties of finite dimensional weak Hopf algebras over a field k. The main results of this paper are:

1. The generalization of the Larson–Sweedler theorem [10] to WHAs claiming that a finite dimensional weak bialgebra is a weak Hopf algebra if and only if it possesses a non-degenerate left integral.
2. The characterization of inequivalent invertible modules of WHAs through left/right grouplike elements in the dual WHA and the proof of the semisimplicity of invertible modules, which include the unit module serving as a monoidal unit in the monoidal category of left (right) modules.
3. A finiteness claim about the order of the antipode (up to an inner automorphism by a grouplike element in the trivial subalgebra) and the derivation of the Radford formula [15] in a weak Hopf algebra A: $S^4(a) = \sigma \rightharpoonup s^{-1}as \leftharpoonup \hat{S}^{-1}(\sigma),\ a \in A$, where S ($\hat{S}$) is the antipode in A ($\hat{A}$), and s and σ are distinguished left grouplike elements in A and in the dual WHA $\hat{A}$, respectively.

We note that it was established in [2] that WHAs are quasi-Frobenius algebras. Result 1 implies that they are Frobenius algebras. Grouplike elements in a WHA were introduced in [2], the modules associated with them were studied in [13]. However, this notion of grouplike elements in a WHA is too restrictive, for characterization of invertible modules in Result 2 one has to introduce the less restrictive notion of left/right grouplike elements. Result 3 was proved in [13] for the special case when the square of the antipode is the identity mapping on the left subalgebra A^L of the WHA A.

The organization of the paper is as follows. In Section 1 we review the axioms and the main properties of weak bialgebras (WBA) and weak Hopf algebras. Here and throughout the paper they are considered to be finite dimensional. Section 2 is devoted to the autonomous monoidal category of modules of a WHA and to properties of the unit module including semisimplicity. In Section 3 we prove a structure theorem for multiple weak Hopf modules and establish the semisimplicity of the modules spanned by integrals of a WHA.

Section 4 contains the generalization of the Larson–Sweedler theorem to the weak Hopf case. In Section 5 we reveal the connection between invertible modules of a WHA A and left/right grouplike elements in the dual WHA $\hat{A}$ and prove that invertible modules are semisimple. Section 6 contains the definition and some basic properties of distinguished left/right grouplike elements, the derivations of the form of the Nakayama automorphism $\theta_\lambda\colon A \to A$ corresponding to a non-degenerate left integral $\lambda \in \hat{A}$ and the Radford formula. We prove also here the already mentioned claim about the order of the antipode and unimodularity of the double of a WHA. In Appendix A we give a simple example of a WHA in which the order of the antipode is not finite. Finally, Appendix B contains the generalization of the cyclic module to weak Hopf algebras containing a modular pair of grouplike elements in involution.

1. Preliminaries

Here we give a quick survey of weak bialgebras and weak Hopf algebras [2]. We restrict ourselves to their main properties, however, some useful identities we use later on are also given.

1.1 The axioms

A *weak bialgebra* $(A; u, \mu; \varepsilon, \Delta)$ is defined by the properties i–iii):

i) A is a finite dimensional associative algebra over a field k with multiplication $\mu\colon A \otimes A \to A$ and unit $u\colon k \to A$, which are k-linear maps.

ii) A is a coalgebra over k with comultiplication $\Delta\colon A \to A \otimes A$ and counit $\varepsilon\colon A \to k$, which are k-linear maps.

iii) The algebra and coalgebra structures obey the compatibility conditions

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad a, b \in A \tag{1.1a}$$

$$\varepsilon(ab^{(1)})\varepsilon(b^{(2)}c) = \varepsilon(abc) = \varepsilon(ab^{(2)})\varepsilon(b^{(1)}c), \qquad a, b, c \in A \tag{1.1b}$$

$$\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}\mathbf{1}^{(1')} \otimes \mathbf{1}^{(2')} = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \otimes \mathbf{1}^{(3)} = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(1')}\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2')}, \tag{1.1c}$$

where (and later on) $ab \equiv \mu(a, b), \mathbf{1} := u(1)$ and we used Sweedler notation [17] for iterated coproducts omitting summation indices and a summation symbol.

A *weak Hopf algebra* $(A; u, \mu; \varepsilon, \Delta; S)$ is a WBA together with property iv):

iv) There exists a k-linear map $S\colon A \to A$, called the antipode, satisfying

$$a^{(1)}S(a^{(2)}) = \varepsilon(\mathbf{1}^{(1)}a)\mathbf{1}^{(2)}, \tag{1.2a}$$

$$S(a^{(1)})a^{(2)} = \mathbf{1}^{(1)}\varepsilon(a\mathbf{1}^{(2)}), \qquad a \in A. \tag{1.2b}$$

$$S(a^{(1)})a^{(2)}S(a^{(3)}) = S(a), \tag{1.2c}$$

WBAs and WHAs are self-dual notions, the dual space $\hat{A} := \mathrm{Hom}_k(A, k)$ of a WBA (WHA) equipped with structure maps $\hat{u}, \hat{\mu}, \hat{\varepsilon}, \hat{\Delta}, (\hat{S})$ defined by transposing the structure maps of A by means of the canonical pairing $\langle\ ,\ \rangle\colon \hat{A} \times A \to k$ gives rise to a WBA (WHA).

1.2 Properties of WBAs

Let A be a WBA. The images $A^{L/R} = \Pi^{L/R}(A) = \bar{\Pi}^{L/R}(A)$ of the projections $\Pi^{L/R}: A \to A$ and $\bar{\Pi}^{L/R}: A \to A$ defined by

$$\begin{aligned} \Pi^L(a) &:= \varepsilon(\mathbf{1}^{(1)}a)\mathbf{1}^{(2)}, & \Pi^R(a) &:= \mathbf{1}^{(1)}\varepsilon(a\mathbf{1}^{(2)}), \\ \bar{\Pi}^L(a) &:= \varepsilon(a\mathbf{1}^{(1)})\mathbf{1}^{(2)}, & \bar{\Pi}^R(a) &:= \mathbf{1}^{(1)}\varepsilon(\mathbf{1}^{(2)}a), \end{aligned} \qquad a \in A \tag{1.3}$$

are unital subalgebras (i.e. containing $\mathbf{1}$) of A that commute with each other. A^L and A^R are called *left* and *right subalgebras*, respectively. The image $\Delta(\mathbf{1})$ of the unit is in $A^R \otimes A^L$ and the coproduct on $A^{L/R}$ reads as:

$$\Delta(x^L) = \mathbf{1}^{(1)}x^L \otimes \mathbf{1}^{(2)}, \quad x^L \in A^L, \qquad \Delta(x^R) = \mathbf{1}^{(1)} \otimes x^R\mathbf{1}^{(2)}, \quad x^R \in A^R. \tag{1.4}$$

Hence, A^L and A^R are left and right coideals, respectively, and the *trivial subalgebra* $A^T := A^L \vee A^R \subset A$ generated by A^L and A^R is a subWBA of A.

The maps $\kappa_L: A^L \to \hat{A}^R$ and $\kappa_R: A^R \to \hat{A}^L$ given by Sweedler arrows

$$\kappa_L(x^L) := x^L \rightharpoonup \hat{\mathbf{1}}, \qquad \kappa_R(x^R) := \hat{\mathbf{1}} \leftharpoonup x^R, \quad x^{L/R} \in A^{L/R} \tag{1.5}$$

are algebra isomorphisms with inverses $\hat{\kappa}_R$ and $\hat{\kappa}_L$, respectively. Moreover, for all $\varphi \in \hat{A}, x^L \in A^L, x^R \in A^R$

$$\begin{aligned} x^L \rightharpoonup \varphi &= (x^L \rightharpoonup \hat{\mathbf{1}})\varphi, & \varphi \leftharpoonup x^L &= (\hat{\mathbf{1}} \leftharpoonup x^L)\varphi, \\ x^R \rightharpoonup \varphi &= \varphi(x^R \rightharpoonup \hat{\mathbf{1}}), & \varphi \leftharpoonup x^R &= \varphi(\hat{\mathbf{1}} \leftharpoonup x^R). \end{aligned} \tag{1.6}$$

The restrictions of the canonical pairing to $\hat{A}^{L/R} \times A^{L/R}$ (four possibilities) are non-degenerate. The maps $\Pi^{L/R}$ and $\hat{\Pi}^{L/R}$ ($\bar{\Pi}^{L/R}$ and $\hat{\bar{\Pi}}^{R/L}$) are transposed to each other

$$\begin{aligned} \langle \varphi, \Pi^{L/R}(a) \rangle &= \langle \hat{\Pi}^{L/R}(\varphi), a \rangle, \\ \langle \varphi, \bar{\Pi}^{L/R}(a) \rangle &= \langle \hat{\bar{\Pi}}^{R/L}(\varphi), a \rangle, \end{aligned} \qquad a \in A, \varphi \in \hat{A}. \tag{1.7}$$

and due to (1.1c) they obey the identities

$$\begin{aligned} \Pi^R(a^{(1)}) \otimes a^{(2)} &= \mathbf{1}^{(1)} \otimes a\mathbf{1}^{(2)}, & \bar{\Pi}^R(a^{(1)}) \otimes a^{(2)} &= \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}a, \\ a^{(1)} \otimes \Pi^L(a^{(2)}) &= \mathbf{1}^{(1)}a \otimes \mathbf{1}^{(2)}, & a^{(1)} \otimes \bar{\Pi}^L(a^{(2)}) &= a\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}. \end{aligned} \qquad a \in A. \tag{1.8}$$

Defining $Z^{L/R} := A^{L/R} \cap \text{Center}\, A$ and $Z := A^L \cap A^R$ the restrictions of $\kappa_{L/R}$ to $Z^{L/R}$ and Z lead to the algebra isomorphisms $Z^{L/R} \to \hat{Z}$ and $Z \to \hat{Z}^{R/L}$, respectively. Hence the *hypercenter* $H := Z \cap \text{Center}\, A = Z^L \cap Z = Z^R \cap Z$ of A is isomorphic to the hypercenter $\hat{H}$ of $\hat{A}$ via the restriction of κ_L or κ_R to H.

The space of *left/right integrals* $I^{L/R}$ in A is defined by

$$I^L := \{l \in A \,|\, al = \Pi^L(a)l, a \in A\}, \qquad I^R := \{r \in A \,|\, ra = r\Pi^R(a), a \in A\}. \tag{1.9}$$

1.3 Properties of WHAs

Let A be a WHA. The antipode S, similarly to the case of Hopf algebras, turns out to be invertible, antimultiplicative, anticomultiplicative and leaves the counit invariant: $\varepsilon = \varepsilon \circ S$. The projections to left and right subalgebras can be expressed as

$$\begin{aligned} \Pi^L(a) &= a^{(1)}S(a^{(2)}), & \Pi^R(a) &= S(a^{(1)})a^{(2)}, \\ \bar{\Pi}^L(a) &= S^{-1}(a^{(2)})a^{(1)}, & \bar{\Pi}^R(a) &= a^{(2)}S^{-1}(a^{(1)}), \end{aligned} \qquad a \in A. \tag{1.10}$$

The restriction of the antipode to A^L leads to algebra antiisomorphism $S: A^L \to A^R$, therefore A^T is a subWHA of A, moreover,

$$\Pi^L \circ S = \Pi^L \circ \Pi^R = S \circ \Pi^R, \qquad \Pi^R \circ S = \Pi^R \circ \Pi^L = S \circ \Pi^L. \tag{1.11}$$

The left and right subalgebras become separable k-algebras with separating idempotents [14] $q^L = S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} \in A^L \otimes A^L$ and $q^R = \mathbf{1}^{(1)} \otimes S(\mathbf{1}^{(2)}) \in A^R \otimes A^R$, respectively, that obey

$$\begin{aligned} x^L S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} &= S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} x^L, & x^L &\in A^L, \\ x^R \mathbf{1}^{(1)} \otimes S(\mathbf{1}^{(2)}) &= \mathbf{1}^{(1)} \otimes S(\mathbf{1}^{(2)}) x^R, & x^R &\in A^R \end{aligned} \tag{1.12}$$

by definition. The product $q^L q^R \in A^T \otimes A^T$ is a separating idempotent for A^T, thus the trivial subalgebra is a separable k-algebra, too. The separating idempotent $q^{L/R}$ serves as a *quasibasis* [20] for the counit:

$$\begin{aligned} S(\mathbf{1}^{(1)})\varepsilon(\mathbf{1}^{(2)}x^L) &= x^L = \varepsilon(x^L S(\mathbf{1}^{(1)}))\mathbf{1}^{(2)}, & x^L &\in A^L, \\ \mathbf{1}^{(1)}\varepsilon(S(\mathbf{1}^{(2)})x^R &= x^R = \varepsilon(x^R \mathbf{1}^{(1)})S(\mathbf{1}^{(2)}), & x^R &\in A^R, \end{aligned} \tag{1.13}$$

thus the counit is a non-degenerate functional on $A^{L/R}$. The properties $S(\mathbf{1}^{(1)})\mathbf{1}^{(2)} = \mathbf{1}$ and $\mathbf{1}^{(1)}S(\mathbf{1}^{(2)}) = \mathbf{1}$ of separating idempotents q_L and q_R ensure that the counit ε is an index $\mathbf{1}$ functional [20] on A^L and on A^R, respectively. Due to the identities (1.5),(1.7) and (1.11–12) the corresponding Nakayama automorphisms $\theta_{L/R}: A^{L/R} \to A^{L/R}$, which are defined by

$$\varepsilon(y^{L/R}\theta_{L/R}(x^{L/R})) := \varepsilon(x^{L/R}y^{L/R}), \qquad x^{L/R}, y^{L/R} \in A^{L/R} \tag{1.14}$$

can be given as

$$\begin{aligned} \theta_L(x^L) &= \mathbf{1} \leftharpoonup \hat{S}^{-1}(\hat{\mathbf{1}} \leftharpoonup x^L) = S^2(x^L), & x^L &\in A^L, \\ \theta_R(x^R) &= \hat{S}(\hat{\mathbf{1}} \leftharpoonup x^R) \rightharpoonup \mathbf{1} = S^{-2}(x^R), & x^R &\in A^R, \end{aligned} \tag{1.15}$$

that is θ_L (θ_R) is the restriction of the square of the (inverse of the) antipode to A^L (A^R). Since any separable algebra admits a non-degenerate (reduced) trace [6] the counit, being a non-degenerate functional on $A^{L/R}$, can be given by the help of the corresponding trace as $\varepsilon(\cdot) = \mathrm{tr}_{L/R}(t_{L/R}\cdot)$ with $t_{L/R} \in A^{L/R}$ invertible. Therefore the Nakayama automorphisms $\theta_{L/R}$ are given by $\mathrm{ad}\, t_{L/R}$ and S^2 is inner on $A^{L/R}$, hence, on A^T, too.

In a WHA a left integral $l \in I^L$ and a right integral $r \in I^R$ obey the identities

$$l^{(1)} \otimes a l^{(2)} = S(a) l^{(1)} \otimes l^{(2)}, \qquad r^{(1)} a \otimes r^{(2)} = r^{(1)} \otimes r^{(2)} S(a), \qquad a \in A, \tag{1.16}$$

respectively. Moreover, there exist projections $L/R\colon A \to I^{L/R}$ and $\bar{L}/\bar{R}\colon A \to I^{L/R}$:

$$\begin{aligned} L(a) &:= \hat{S}^2(\beta_i) \rightharpoonup (b_i a), \qquad & R(a) &:= (a b_i) \leftharpoonup \hat{S}^2(\beta_i), \\ \bar{L}(a) &:= (b_i a) \leftharpoonup \hat{S}^{-2}(\beta_i), \qquad & \bar{R}(a) &:= \hat{S}^{-2}(\beta_i) \rightharpoonup (a b_i), \end{aligned} \qquad a \in A \tag{1.17}$$

where $\{b_i\} \subset A$ and $\{\beta_i\} \subset \hat{A}$ are dual k-bases with respect to the canonical pairing. They obey the properties

$$\begin{aligned} \langle \hat{L}/\hat{R}(\varphi), a \rangle &= \langle \varphi, R/L(a) \rangle, \\ \langle \hat{\bar{L}}/\hat{\bar{R}}(\varphi), a \rangle &= \langle \varphi, \bar{L}/\bar{R}(a) \rangle, \end{aligned} \qquad a \in A, \varphi \in \hat{A}, \tag{1.18}$$

therefore the restrictions of the canonical pairing to $\hat{I}^{L/R} \times I^{L/R}$ (four possibilities) are non-degenerate.

2. Properties of the unit modules

In this Chapter A denotes a WHA over a field k.

A *left (right) A-module* ${}_AM \equiv (M, \mu_L)$ ($M_A \equiv (M, \mu_R)$) is a k-linear space together with the k-linear map $\mu_L\colon A \otimes M \to M$ ($\mu_R\colon M \otimes A \to M$) satisfying

$$\begin{aligned} a \cdot (b \cdot m) &= (ab) \cdot m, \qquad & (m \cdot a) \cdot b &= m \cdot (ab), \\ \mathbf{1} \cdot m &= m, \qquad & m \cdot \mathbf{1} &= m, \end{aligned} \qquad m \in M, a, b \in A,$$

where (and later on) $\mu_L(a \otimes m) \equiv a \cdot m$ and $\mu_R(m \otimes a) \equiv m \cdot a$.

Definition 2.1 *The unit left (right) A-module ${}_AA^L$ (A^R_A) is defined by*

$$\begin{aligned} a \cdot x^L &:= \Pi^L(a x^L) = a^{(1)} x^L S(a^{(2)}), \qquad & x^L \in A^L, a \in A \\ x^R \cdot a &:= \Pi^R(x^R a) = S(a^{(1)}) x^R a^{(2)}, \qquad & x^R \in A^R, a \in A. \end{aligned} \tag{2.1}$$

We note that these modules need not be one-dimensional as in the case of Hopf algebras, they are not even irreducible in general. Nevertheless, they play the role of the unit object in the monoidal category of finite dimensional left (right) A-modules. Here we deal with only the category of left A-modules, the case of right A-modules can be treated similarly.

Proposition 2.2 *The category $\mathcal{L}$ consisting of finite dimensional left A-modules of a WHA A as objects and left A-module maps as arrows can be endowed with an autonomous (relaxed) monoidal structure: $(\mathcal{L}; \times, {}_AA^L, \{1_{K\times M\times N}\}, \{X^L_M\}, \{X^R_M\}; \overleftharpoon{\ }, \overrightharpoon{\ })$, where $\times$ is the monoidal product, ${}_AA^L$ is the monoidal unit, $\{1_{K\times M\times N}\}, \{X^L_M\}, \{X^R_M\}$ are natural* *equivalences satisfying the pentagon and the triangle identities, while* $\overleftarrow{}$ *and* $\overrightarrow{}$ *are the functors of left and right conjugations, respectively.*

Proof. Let us define first the monoidal product: $\times$. The product module ${}_A(M \times N)$ of the modules ${}_AM$ and ${}_AN$ as a k-linear space is defined to be

$$M \times N := \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} \cdot N \tag{2.2a}$$

and the left A-module structure on $M \times N$ is given by

$$a \cdot (m \otimes n) := a^{(1)} \cdot m \otimes a^{(2)} \cdot n, \qquad a \in A,\ m \otimes n \in M \times N, \tag{2.2b}$$

where (and later on) we have suppressed possible or necessary summation for tensor product elements in product modules. The product on the arrows $T_\alpha \colon M_\alpha \to N_\alpha, \alpha = 1, 2$ is defined by $T_1 \times T_2 := (T_1 \otimes T_2) \circ \Delta(\mathbf{1})$, i.e. by the restriction of the tensor product of the linear maps T_1 and T_2 to $M_1 \times M_2$. One can easily check that $T_1 \times T_2 \colon M_1 \times M_2 \to N_1 \times N_2$ is a left A-module map. The given monoidal product is associative due to the associativity of the coproduct and property (1.1c) of the unit, hence the components $M_1 \times (M_2 \times M_3) \to (M_1 \times M_2) \times M_3$ of the natural equivalence responsible for associativity in a monoidal category are the identity mappings $1_{M_1 \times M_2 \times M_3}$ in our case.

The monoidal unit property of the left A-module A^L can be seen by verifying that for any object M the k-linear invertible maps $X^L_M \colon M \to A^L \times M$ and $X^R_M \colon M \to M \times A^L$ defined by

$$\begin{aligned} X^L_M(m) &:= S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} \cdot m, \qquad & X^R_M(m) &:= \mathbf{1}^{(1)} \cdot m \otimes \mathbf{1}^{(2)}, \\ X^{L-1}_M(x^L \otimes m) &:= x^L \cdot m, & X^{R-1}_M(m \otimes x^L) &:= S^{-1}(x^L) \cdot m, \end{aligned} \tag{2.3}$$

are left A-module maps and the identities

$$\begin{aligned} X^L_N T &= (1_{A^L} \times T) X^L_M \\ X^R_N T &= (T \times 1_{A^L}) X^L_M \end{aligned} \qquad M, N \in \mathrm{Obj}\, \mathcal{L}, \quad T \colon M \to N, \tag{2.4}$$

$$(X^R_M \times 1_N)(1_M \times X^{L-1}_N) = 1_{M \times A^L \times N} \tag{2.5}$$

hold, i.e. $X^L = \{X^L_M\}$ and $X^R = \{X^R_M\}$ are natural equivalences satisfying the triangle identity.

An autonomous category [21] contains both left and right conjugation functors by definition. The left conjugate $\overleftarrow{M}$ of an object M in $\mathcal{L}$ is given by the k-dual $\hat{M} := \mathrm{Hom}_k(M, k)$ as a k-linear space. The left A-module structure $\overleftarrow{M} \equiv (\hat{M}, \overleftarrow{\mu}_L)$ is defined to be

$$\langle a \cdot \hat{m}, m \rangle := \langle \hat{m}, S(a) \cdot m \rangle, \qquad a \in A, m \in M, \hat{m} \in \hat{M}, \tag{2.6}$$

where $\langle\, , \,\rangle$ is the k-valued canonical bilinear paring on the cartesian product of $\hat{M}$ and M. Dual bases with respect to this pairing will be denoted by $\{\hat{m}_i\}_i \subset \hat{M}$ and $\{m_i\}_i \subset M$. Due to the definition (2.6) of the left A-module $\overleftarrow{M}$ we have

$$m_i \otimes a \cdot \hat{m}_i = S(a) \cdot m_i \otimes \hat{m}_i, \quad a \in A \tag{2.7a}$$

$$\begin{aligned} m_i \otimes \hat{m}_i &= \mathbf{1} \cdot m_i \otimes \hat{m}_i = \mathbf{1}^{(1)} S(\mathbf{1}^{(2)}) \cdot m_i \otimes \hat{m}_i \\ &= \mathbf{1}^{(1)} \cdot m_i \otimes \mathbf{1}^{(2)} \cdot \hat{m}_i \in M \times \overleftarrow{M}, \end{aligned} \tag{2.7b}$$

where (and later on) we omit summation symbol for the sum of tensor product of dual basis elements.

The arrow family of left evaluation and coevaluation maps $E^l_M\colon \overleftarrow{M} \times M \to A^L$ and $C^l_M\colon A^L \to M \times \overleftarrow{M}$, respectively, are defined to be

$$\begin{aligned} E^l_M(\hat m \otimes m) &:= \mathbf{1}^{(2)}\langle \hat m, \mathbf{1}^{(1)} \cdot m\rangle, \qquad \hat m \otimes m \in \overleftarrow{M} \times M \\ C^l_M(x^L) &:= x^L \cdot m_i \otimes \hat m_i, \qquad x^L \in A^L. \end{aligned} \tag{2.8}$$

They are left A-module maps

$$\begin{aligned} E^l_M(a\cdot(\hat m \otimes m)) &= \mathbf{1}^{(2)}\langle a^{(1)}\cdot \hat m, \mathbf{1}^{(1)}a^{(2)}\cdot m\rangle = \mathbf{1}^{(2)}\langle \hat m, S(a^{(1)})\mathbf{1}^{(1)}a^{(2)}\cdot m\rangle \\ &= \Pi^L(a^{(3)})\langle \hat m, S(a^{(1)})a^{(2)}\cdot m\rangle = \Pi^L(a^{(2)})\langle \hat m, \Pi^R(a^{(1)})\cdot m\rangle \\ &= \Pi^L(a\mathbf{1}^{(2)})\langle \hat m, \mathbf{1}^{(1)}\cdot m\rangle = a\cdot E^l_M(\hat m\otimes m), \\ C^l_M(a\cdot x^L) &= a^{(1)}x^L S(a^{(2)})\cdot m_i \otimes \hat m_i = a^{(1)}x^L\cdot m_i\otimes a^{(2)}\cdot \hat m_i \\ &= a\cdot C^l_M(x^L) \end{aligned} \tag{2.9}$$

due to the identities (1.8) and (2.7a) and they satisfy the left rigidity identities [21]

$$\begin{aligned} X^{R-1}_M(1_M\times E^l_M)(C^l_M\times 1_M)X^L_M(m) &:= S^{-1}(\mathbf{1}^{(2')})S(\mathbf{1}^{(1)})\cdot m_i\langle \hat m_i, \mathbf{1}^{(1')}\mathbf{1}^{(2)}\cdot m\rangle \\ = S^{-1}(\mathbf{1}^{(2')})S(\mathbf{1}^{(1)})\mathbf{1}^{(1')}\mathbf{1}^{(2)}\cdot m &= m, \qquad m\in M, \end{aligned} \tag{2.10a}$$

$$\begin{aligned} X^{L-1}_{\overleftarrow{M}}(E^l_M\times 1_{\overleftarrow{M}})(1_{\overleftarrow{M}}\times C^l_M)X^R_{\overleftarrow{M}}(\hat m) &:= \langle \mathbf{1}^{(1)}\cdot\hat m, \mathbf{1}^{(1')}\mathbf{1}^{(2)}\cdot m_i\rangle \mathbf{1}^{(2')}\cdot \hat m_i \\ = \mathbf{1}^{(2')}S^{-1}(\mathbf{1}^{(1')}\mathbf{1}^{(2)})\mathbf{1}^{(1)}\cdot \hat m &= \hat m, \qquad \hat m\in \overleftarrow{M} \end{aligned} \tag{2.10b}$$

for any $M\in \operatorname{Obj}\mathcal{L}$. Thus defining the left conjugated arrow $\overleftarrow{T}\colon \overleftarrow{N}\to\overleftarrow{M}$ of $T\colon M\to N$ by

$$\overleftarrow{T} := X^{L-1}_{\overleftarrow{M}}(E^l_N\times 1_{\overleftarrow{M}})(1_{\overleftarrow{N}}\times T\times 1_{\overleftarrow{M}})(1_{\overleftarrow{N}}\times C^l_M)X^R_{\overleftarrow{N}} \tag{2.11}$$

one arrives at the antimonoidal contravariant left conjugation functor $\overleftarrow{\ \ }\colon \mathcal{L}\to\mathcal{L}$ [21].

Similarly, the right conjugate $\overrightarrow{M}$ of an object M in $\mathcal{L}$ is the k-linear space $\hat M$ equipped with the left A-module structure $\overrightarrow{M}\equiv(\hat M, \overrightarrow{\mu}_L)$

$$\langle a\cdot\hat m, m\rangle := \langle \hat m, S^{-1}(a)\cdot m\rangle, \qquad a\in A, m\in M, \hat m\in\hat M \tag{2.12}$$

implying

$$\hat m_i\otimes a\cdot m_i = S(a)\cdot\hat m_i\otimes m_i, \quad a\in A \tag{2.13a}$$

$$\begin{aligned} \hat m_i\otimes m_i &= \mathbf{1}\cdot\hat m_i\otimes m_i = \mathbf{1}^{(1)}S(\mathbf{1}^{(2)})\cdot\hat m_i\otimes m_i \\ &= \mathbf{1}^{(1)}\cdot\hat m_i\otimes\mathbf{1}^{(2)}\cdot m_i\in\overrightarrow{M}\times M. \end{aligned} \tag{2.13b}$$

The arrow family of right evaluation and coevaluation maps $E^r_M: M \times \overrightarrow{M} \to A^L$ and $C^r_M: A^L \to \overrightarrow{M} \times M$, respectively, are defined to be

$$\begin{aligned} E^r_M(m \otimes \hat{m}) &:= \mathbf{1}^{(2)} \langle \mathbf{1}^{(1)} \cdot \hat{m}, m \rangle, \qquad m \otimes \hat{m} \in M \times \overrightarrow{M}, \\ C^r_M(x^L) &:= x^L \cdot \hat{m}_i \otimes m_i, \qquad x^L \in A^L. \end{aligned} \tag{2.14}$$

Similarly to the previous case one proves that they are left A-module maps satisfying the right rigidity identities [21]

$$\begin{aligned} X^{L-1}_M (E^r_M \times 1_M)(1_M \times C^r_M) X^R_M &= 1_M, \\ X^{R-1}_{\overrightarrow{M}} (1_{\overrightarrow{M}} \times E^r_M)(C^r_M \times 1_{\overrightarrow{M}}) X^L_{\overrightarrow{M}} &= 1_{\overrightarrow{M}}. \end{aligned} \tag{2.15}$$

Hence, defining the right conjugated arrow $\overrightarrow{T}: \overrightarrow{N} \to \overrightarrow{M}$ of $T: M \to N$ by

$$\overrightarrow{T} := X^{R-1}_{\overrightarrow{M}} (1_{\overrightarrow{M}} \times E^r_N)(1_{\overrightarrow{M}} \times T \times 1_{\overrightarrow{N}})(C^r_M \times 1_{\overrightarrow{N}}) X^L_{\overrightarrow{N}} \tag{2.16}$$

one arrives at the antimonoidal contravariant right conjugation functor $\overrightarrow{\ \ }: \mathcal{L} \to \mathcal{L}$. ■

In the following we establish the essential properties of the unit module.

Lemma 2.3 *The trivial weak Hopf subalgebra $A^T \subset A$ is a sum of simple subcoalgebras, i.e. A^T is contained in the coradical C_0 of A.*

Proof. First we decompose the WHA A^T into a direct sum of subWHAs.

The intersection $Z := A^L \cap A^R$ is in the center of the separable algebra A^T, because the unital subalgebras A^L and A^R of A^T commute with each other. The WHA identity (1.11) implies $z = S(z)$ for all $z \in Z$. Hence, Z is a unital, pointwise S-invariant subalgebra of the k-algebra Center A^T and one can write A^T as an amalgamated tensor product algebra $A^T \simeq A^L \otimes_Z A^R$.

Let $\{z_\alpha\}_\alpha \subset Z$ be the complete set of primitive idempotents in Z. They are central idempotents in A^T, therefore $A^T = \oplus_\alpha A^T_\alpha$, $A^T_\alpha := A^T z_\alpha$ is an ideal decomposition of the algebra A^T. But it is also a WHA decomposition: first, $S(A^T_\alpha) = A^T_\alpha$, beacause Z is pointwise S-invariant, and second, $\Delta(A^T_\alpha) \subset A^T_\alpha \otimes A^T_\alpha$, because

$$\Delta(x z_\alpha) = \Delta(x z_\alpha z_\alpha) = \Delta(x)(z_\alpha \otimes \mathbf{1})(\mathbf{1} \otimes z_\alpha) = \Delta(x)(z_\alpha \otimes z_\alpha), \quad x \in A^T, \tag{2.17}$$

due to the form (1.4) of the coproduct for elements in A^L and in A^R and due to $z_\alpha \in A^L \cap A^R$.

This decomposition implies that $(A^T_\alpha)^X = A^X_\alpha$ with $X = L, R, T$ and that the WHA A^T_α has the amalgamated tensor product algebra structure $A^T_\alpha \simeq A^L_\alpha \otimes_{Z_\alpha} A^R_\alpha$ with unit $\mathbf{1}_\alpha = z_\alpha$. The algebra $Z_\alpha := Z z_\alpha = A^L_\alpha \cap A^R_\alpha$ is an Abelian division algebra over the ground field k in the center of the separable algebra A^T_α, hence Z_α is a finite separable field extension of k [14].

Now we prove that the dual $\hat{A}^T_\alpha$ of the WHA A^T_α is isomorphic to the simple algebra $M_{n_\alpha}(Z_\alpha)$, where $n_\alpha = \dim_{Z_\alpha} A^L_\alpha$, i.e. A^T_α is simple as a coalgebra. For notational simplicity we omit the α index. Since Z is also a unital subalgebra of the center of A^L, that is A^L can be considered as an algebra over the field Z, there exists a product k-basis $\{e_{\mu i}\}_{\mu,i} \equiv \{v_\mu e_{1i}\}_{\mu,i}$ of A^L, where $\{v_\mu\}_\mu$ is a k-basis of Z with $v_1 = \mathbf{1}$. The identity (1.13) shows that the counit is a non-degenerate functional on A^L due to the separability identity (1.12). Hence, there exists a dual k-basis $\{f_{\mu i}\}_{\mu,i}$ of A^L with respect to the counit: $\varepsilon(e_{\mu i} f_{\nu j}) = \delta_{\mu\nu}\delta_{ij}$. The dual basis also has a product structure: $\{f_{\mu i}\}_{\mu,i} \equiv \{w_\mu f_{1i}\}_{\mu,i}$, because $z \mapsto \varepsilon(z e_{1i} f_{1i})$ defines a nonzero (hence nondegenerate) k-linear map $E\colon Z \to k$ for any i. The set $\{w_\mu\}_\mu \subset Z$ is nothing else that the dual basis of $\{v_\mu\}_\mu$ with respect to E and $w_\mu = \sum_\nu (b^{-1})_{\nu\mu} v_\nu$, where b^{-1} is the inverse of the symmetric matrix b having matrix elements $b_{\mu\nu} := E(v_\mu v_\nu) \in k$. Therefore the separating idempotent $S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} \in A^L \otimes A^L$, which is also a quasibasis [20] of the counit as a nondegenerate functional on A^L, can be given in terms of the dual bases and

$$\begin{aligned} \Delta(\mathbf{1}) = S^{-1}(S(\mathbf{1}^{(1)})) \otimes \mathbf{1}^{(2)} &= \sum_{\mu,i} S^{-1}(w_\mu f_{1i}) \otimes v_\mu e_{1i} \\ &= \sum_{\mu,i} w_\mu S^{-1}(f_{1i}) \otimes v_\mu e_{1i} \in A^R \otimes A^L. \end{aligned} \tag{2.18}$$

Choosing the basis $\{\varphi^\mu_{ij}\}_{i,j,\mu}$ of $\hat{A}^T$ dual to the product basis $\{e_{1i} v_\mu S^{-1}(f_{1j})\}_{i,j,\mu}$ of A^T with respect to the canonical pairing $\hat{A}^T \times A^T \to k$ one can compute the product

$$\begin{aligned} \varphi^\mu_{ij}\varphi^\nu_{kl} &= \sum_{m,n,\lambda} \varphi^\lambda_{mn} \langle \varphi^\mu_{ij} \otimes \varphi^\nu_{kl}, \Delta(e_{1m} v_\lambda S^{-1}(f_{1n}))\rangle \\ &= \sum_{m,n,\lambda} \varphi^\lambda_{mn} \langle \varphi^\mu_{ij} \otimes \varphi^\nu_{kl}, \Delta(\mathbf{1})(e_{1m} \otimes v_\lambda S^{-1}(f_{1n}))\rangle \\ &= \sum_{\substack{m,n,\lambda \\ p,\rho}} \varphi^\lambda_{mn} \langle \varphi^\mu_{ij} \otimes \varphi^\nu_{kl}, e_{1m} w_\rho f_{1p} \otimes e_{1p} v_\rho v_\lambda S^{-1}(f_{1n}))\rangle \\ &= \delta_{jk} \sum_\lambda (\sum_\rho (b^{-1})_{\mu\rho} c^\nu_{\rho\lambda}) \varphi^\lambda_{il}, = \delta_{jk} \sum_\lambda \bar{c}^\lambda_{\mu\nu} \varphi^\lambda_{il}, \end{aligned} \tag{2.19}$$

where $\{c^\lambda_{\mu\nu}\}$ and $\{\bar{c}^\lambda_{\mu\nu} = \sum_\rho (b^{-1})_{\mu\rho} c^\nu_{\rho\lambda}\}$ are the k-valued structure coefficients of the algebra Z in the dual bases: $v_\mu v_\nu = \sum_\lambda c^\lambda_{\mu\nu} v_\lambda$ and $w_\mu w_\nu = \sum_\lambda \bar{c}^\lambda_{\mu\nu} w_\lambda$. Hence, the elements of the basis $\{\varphi^\mu_{ij}\}_{i,j,\mu} \subset \hat{A}^T$ are just the matrix units of $M_n(Z)$ over k. ∎

Theorem 2.4 *The unit left A-module ${}_A A^L$ is a direct sum of irreducible submodules*

$$ {}_A A^L = \bigoplus_p {}_A A^L_p, \qquad A^L_p := A^L z^L_p, \tag{2.20}$$

where $\{z^L_p\}_p$ is the complete set of primitive idempotents in $Z^L := A^L \cap \text{Center}\, A$.

Proof. Let N be the radical of A. Since N is an ideal in A we have $N \cdot A^L = \Pi^L(NA^L) \subset \Pi^L(N)$. But due to the identity $N = (\hat{C}_0)^\perp := \{a \in A \,|\, \langle \hat{C}_0, a\rangle = 0\}$ [17], where $\hat{C}_0$ is the coradical of the dual weak Hopf algebra $\hat{A}$, the previous Lemma leads to the containment $N = (\hat{C}_0)^\perp \subset (\hat{A}^T)^\perp \subset (\hat{A}^L)^\perp$, hence using (1.7) the canonical pairing gives rise to

$$\langle \hat{A}, \Pi^L(N)\rangle = \langle \hat{\Pi}^L(\hat{A}), N\rangle = \langle \hat{A}^L, N\rangle = 0, \tag{2.21}$$

i.e. $\Pi^L(N) = 0$. Therefore the radical of A is in the annihilator ideal of the left module ${}_AA^L$, that is ${}_AA^L$ is completely reducible.

Now we prove that $\mathrm{End}\,{}_AA^L = Z^L\cdot$, that is the restriction of the A-action to the subalgebra Z^L on A^L. The $\mathrm{End}\,{}_AA^L \supset Z^L\cdot$ containment is trivial. For the opposite containment one notes that any $f \in \mathrm{End}\,{}_AA^L$ can be characterized by right multiplication by $f(\mathbf{1}) \in A^L$, because for all $x^L \in A^L$

$$f(x^L) = f(x^L\mathbf{1}) = f(\Pi^L(x^L\mathbf{1})) = f(x^L \cdot \mathbf{1}) = x^L \cdot f(\mathbf{1}) = \Pi^L(x^L f(\mathbf{1})) = x^L f(\mathbf{1}). \tag{2.22}$$

Hence,

$$\begin{aligned}\Pi^L(af(\mathbf{1})) &= a \cdot f(\mathbf{1}) = f(a \cdot \mathbf{1}) = f(\Pi^L(a)) = f(\Pi^L(a)\cdot \mathbf{1}) = \Pi^L(a)\cdot f(\mathbf{1})\\ &= \Pi^L(\Pi^L(a)f(\mathbf{1})) = \Pi^L(a)f(\mathbf{1}), \qquad a \in A,\end{aligned} \tag{2.23}$$

therefore using (1.2c), (1.10), (2.23) and that the subalgebras A^L and A^R commute with each other

$$\begin{aligned}f(\mathbf{1})S(a) &= f(\mathbf{1})S(a^{(1)})a^{(2)}S(a^{(3)}) = S(a^{(1)})a^{(2)}f(\mathbf{1})S(a^{(3)})\\ &= S(a^{(1)})\Pi^L(a^{(2)}f(\mathbf{1})) = S(a^{(1)})\Pi^L(a^{(2)})f(\mathbf{1}) = S(a)f(\mathbf{1}), \quad a \in A,\end{aligned} \tag{2.24}$$

implies that $f(\mathbf{1}) \in A^L \cap \mathrm{Center}\,A =: Z^L$, hence (2.22) leads to $f(x^L) = x^L f(\mathbf{1}) = f(\mathbf{1})x^L = f(\mathbf{1}) \cdot x^L$.

Now, the direct sum decomposition (2.20) is clear and $\mathrm{End}\,{}_AA^L_p = Z^L z^L_p\cdot$. But $Z^L z^L_p$ is an (Abelian) division algebra, therefore ${}_AA^L_p$ is indecomposable [5]. Together with complete reducibility of the unit module ${}_AA^L$ this leads to the irreducibility of the direct summands ${}_AA^L_p$. ■

Remark 2.5 Similarly one can prove that the unit right A-module A^R_A given in Def. 2.1 is a direct sum of irreducible submodules

$$A^R_A = \bigoplus_p A^R_{pA}, \qquad A^R_p := A^R z^R_p \tag{2.25}$$

where $\{z^R_p := S(z^L_p)\}_p$ is the complete set of primitive idempotents in $Z^R := A^R \cap \mathrm{Center}\,A$. ■

The product of primitive idempotents in Z^L and Z^R gives rise to a decomposition of the unit $\mathbf{1} = \sum_{p,q} z^L_p z^R_q \equiv \sum_{p,q} z^L_p S(z^L_q)$ in $Z^L \vee Z^R \subset A^T \cap \mathrm{Center}\,A$.[1] Of course,

[1] Note that $S^2(z^{L/R}_p) = z^{L/R}_p$, because S^2 is inner on $A^{L/R}$ and the idempotents are central.

certain products are identically zero due to the amalgamation with the hypercenter $H = A^L \cap A^R \cap \mathrm{Center}\, A$, because $Z^L \vee Z^R \simeq Z^L \otimes_H Z^R$. If $z^L_p z^R_q \neq 0$ we refer to (p,q) as an *admissible pair*. Hence, the non-zero summands are labelled by admissible pairs in the decomposition of the unit, which induces a direct sum decomposition of every A-module ${}_AM$ into submodules as $M = \oplus_{(p,q)} M_{(p,q)}$, where $M_{(p,q)} := z^L_p z^R_q \cdot M$. We will call the left A-module ${}_AM$ a *member of the* (p,q) *class* and write ${}_AM_{(p,q)}$ if (p,q) is an admissible pair and

$$z^L_{p'} z^R_{q'} \cdot M = \delta_{pp'} \delta_{qq'} M, \tag{2.26}$$

for all product idempotents $z^L_{p'} z^R_{q'} \in Z^L \vee Z^R$. Clearly, the irreducible submodule ${}_AA^L_p$ of the unit module ${}_AA^L$ is in the class (p,p), because

$$z^L_q z^R_r \cdot x z^L_p := z^L_q x z^L_p S(z^R_r) = z^L_q x z^L_p z_r L = \delta_{p,q} \delta_{p,r} x z^L_p, \qquad x \in A^L. \tag{2.27}$$

The next Lemma shows that the irreducible submodules of the unit module ${}_AA^L$ obey a kind of minimality condition in the corresponding class of left A-modules.

Lemma 2.6 *i) The k-dimension* $|M_{(p,q)}|$ *of a nonzero left A-module* ${}_AM_{(p,q)}$ *in the* (p,q) *class obeys the inequality*

$$|M_{(p,q)}| \geq \max\{|A^L_p|, |A^L_q|\}. \tag{2.28}$$

ii) The restriction of A to the subalgebras A^L_p *and* A^R_q *makes* ${}_AM_{(p,q)}$ *a faithful left* A^L_p*- and* A^R_q*-modules, respectively.*

Proof. Using property (1.12) of the separating idempotent of A^L and the decomposition of the unit into primitive idempotens in Z^L one obtains

$$\Delta(\mathbf{1}) = \sum_r S^{-1}(S(\mathbf{1}^{(1)})) \otimes \mathbf{1}^{(2)} z^L_r = \sum_r (z^R_r \otimes z^L_r) \Delta(\mathbf{1}). \tag{2.29}$$

Therefore, for any two left A-modules M, N within a certain class we have

$$\begin{aligned} z^L_p z^R_q \cdot M_{(p_1,q_1)} \times N_{(p_2,q_2)} &= z^L_p z^R_q \cdot (\Delta(\mathbf{1})(M_{(p_1,q_1)} \otimes N_{(p_2,q_2)})) \\ &= \sum_r z^L_p z^R_r \mathbf{1}^{(1)} \cdot M_{(p_1,q_1)} \otimes z^L_r z^R_q \mathbf{1}^{(2)} \cdot N_{(p_2,q_2)}, \end{aligned} \tag{2.30}$$

implying

$${}_AM_{(p_1,q_1)} \times {}_AN_{(p_2,q_2)} = \begin{cases} 0, & q_1 \neq p_2, \\ {}_A(M \times N)_{(p_1,q_2)}, & q_1 = p_2. \end{cases} \tag{2.31}$$

In the proof of Lemma 2.3 we have seen that the separating idempotent of A^L has the expression $S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} = \sum_i f_i \otimes e_i$, where $\{e_i\}_i, \{f_i\}_i \subset A^L$ are dual bases with respect to the counit: $\varepsilon(e_i f_j) = \delta_{i,j}$. Choosing a basis $\{e_i\}_i = \cup_p \{e_i\}_{i\in p}$ that respects the direct sum decomposition $A^L = \oplus_p A^L_p$, i.e. $\{e_i\}_{i\in p} \subset A^L_p$ then $f_i \otimes e_i \in A^L_p \otimes A^L_p, i \in p$ follows. Since X^L_M and X^R_M defined in (2.3) are left A-module isomorphisms

$$\begin{aligned} |M_{(p,q)}| &= |A^L \times M_{(p,q)}| = |S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} z^L_p \cdot M_{(p,q)}| \\ &= |\sum_{i\in p} f_i \otimes e_i \cdot M_{(p,q)}| = \sum_{i\in p} |e_i \cdot M_{(p,q)}|, \end{aligned} \tag{2.32a}$$

$$
\begin{aligned}
|M_{(p,q)}| &= |M_{(p,q)} \times A^L| = |\mathbf{1}^{(1)} z^R_q \cdot M_{(p,q)} \otimes \mathbf{1}^{(2)}| \\
&= |\sum_{i\in q} S^{-1}(f_i) \cdot M_{(p,q)} \otimes e_i| = \sum_{i\in q} |S^{-1}(f_i) \cdot M_{(p,q)}|
\end{aligned}
\tag{2.32b}
$$

for any left A-module $M_{(p,q)}$ in the (p,q) class. Hence, if we prove that $M_{(p,q)}$ is a faithful left A^L_p- and A^R_q-module, i.e. $x^L_p \cdot M_{(p,q)}$ and $x^R_q \cdot M_{(p,q)}$ are nonzero linear subspaces of $M_{(p,q)}$ for all non-zero elements $x^L_p \in A^L_p$ and $x^R_q \in A^R_q$, respectively, then we are ready, because a nonzero linear subspace is at least one dimensional and $|A^R_q| = |S(A^R_q)| = |A^L_q|$ due to the invertibility of the antipode S.

Let us suppose that $0 \neq x^L_p \in A^L_p$ is in the annihilator ideal of ${}_AM_{(p,q)}$. Since in the irreducible left A-module ${}_AA^L_p$ every non-zero vector is cyclic $A^L_p = \{a \cdot x^L_p := a^{(1)} x^L_p S(a^{(2)}) \,|\, a \in A\}$ should also be contained in the annihilator ideal of ${}_AM_{(p,q)}$. But this contradicts to the assumption that ${}_AM_{(p,q)}$ is a nonzero module in the (p,q) class. Using the irreducibility of the right A-module A^R_{qA} (see Remark 2.5) one has $A^R_{qA} = \{x^R_q \cdot a := S(a^{(1)}) x^R_q a^{(2)} \,|\, a \in A\}$ for any non-zero $x^R_q \in A^R_q$ and the assumption that a non-zero element of A^R_q is in the annihilator ideal of ${}_AM_{(p,q)}$ leads to the contradiction as before. ■

3. Hopf modules in weak Hopf algebras

Besides A-modules we need the notion of weak Hopf modules of a WBA A [2]. First, a *left (right) A-comodule* is a pair ${}^AM \equiv (M, \delta_L)$ ($M^A \equiv (M, \delta_R)$) consisting of a finite dimensional k-linear space and a k-linear map $\delta_L \colon M \to A \otimes M$ ($\delta_R \colon M \to M \otimes A$) called the coaction that obeys

$$
\begin{aligned}
(id_A \otimes \delta_L) \circ \delta_L &= (\Delta \otimes id_M) \circ \delta_L, & (\delta_R \otimes id_A) \circ \delta_R &= (id_M \otimes \Delta) \circ \delta_R, \\
(\varepsilon \otimes id_M) \circ \delta_L &= id_M, & (id_M \otimes \varepsilon) \circ \delta_R &= id_M.
\end{aligned}
\tag{3.1}
$$

They incorporate only the coalgebra properties of A. In the following we will use the notations $\delta_L(m) \equiv m_{-1} \otimes m_0$ and $\delta_R(m) \equiv m_0 \otimes m_1$. Lower and upper A-indices will indicate A-modules and A-comodules, respectively.

The *weak Hopf modules* (WHM) $M^A_A, {}_AM^A, {}^A_AM, {}^AM_A$ of a WBA A are A-modules and A-comodules simultaneously together with a compatibility condition restricting the comodule map to be an A-module map, e.g.

$$
\begin{aligned}
M^A_A \equiv (M, \mu_R, \delta_R): &\quad (m \cdot a)_0 \otimes (m \cdot a)_1 = m_0 \cdot a^{(1)} \otimes m_1 a^{(2)}, \\
{}_AM^A \equiv (M, \mu_L, \delta_R): &\quad (a \cdot m)_0 \otimes (a \cdot m)_1 = a^{(1)} \cdot m_0 \otimes a^{(2)} m_1,
\end{aligned}
\qquad a \in A, m \in M.
\tag{3.2}
$$

As a consequence of these identities WHMs obey a kind of non-degeneracy property

$$
\begin{aligned}
m &= m_0 \cdot \Pi^R(m_1), \quad m \in M^A_A; & m &= \bar\Pi^R(m_1) \cdot m_0, \quad m \in {}_AM^A; \\
m &= m_0 \cdot \bar\Pi^L(m_{-1}), \quad m \in {}^AM_A; & m &= \Pi^L(m_{-1}) \cdot m_0, \quad m \in {}^A_AM.
\end{aligned}
\tag{3.3}
$$

We call ${}_AM_A^A, {}^AM_A^A, {}_A^AM_A, {}_A^AM^A, {}_A^AM_A^A$ *multiple weak Hopf modules* if they are pairwise WHMs of the WBA A in the possible A-indices and if the different module or comodule maps commute, e.g.

$$\begin{aligned} {}_AM_A^A &\equiv (M,\mu_R,\mu_L,\delta_R): &\quad \mu_R\circ(\mu_L\otimes id_A) &= \mu_L\circ(id_A\otimes\mu_R), \\ {}^AM_A^A &\equiv (M,\mu_R,\delta_L,\delta_R): &\quad (\delta_L\otimes id_A)\circ\delta_R &= (id_A\otimes\delta_R)\circ\delta_L. \end{aligned} \tag{3.4}$$

The *invariants* and *coinvariants* of left/right A-modules and left/right A-comodules, respectively, are defined to be

$$\begin{aligned} I({}_AM) &:= \{m\in M\,|\,a\cdot m=\pi^L(a)\cdot m, a\in A\}, \quad C({}^AM) := \{m\in M\,|\,\delta_L(m)\in A^R\otimes M\}, \\ I(M_A) &:= \{m\in M\,|\,m\cdot a=m\cdot\pi^R(a), a\in A\}, \quad C(M^A) := \{m\in M\,|\,\delta_R(m)\in M\otimes A^L\}. \end{aligned} \tag{3.5}$$

For example, the left/right invariants and the left/right coinvariants of the multiple weak Hopf module ${}_A^AA_A^A \equiv (A,\mu,\mu,\Delta,\Delta)$ of a WBA A are the left/right integrals $I^{L/R}$ and the right/left subalgebras $A^{R/L}$, respectively. Dualizing left/right actions or coactions of a WBA A by the help of dual bases in A and $\hat{A}$ with respect to the canonical pairing one arrives at right/left coactions or actions of the dual WBA $\hat{A}$, respectively, e.g.

$$M^{\hat{A}}: \qquad \hat{\delta}_R(m) := b_i\cdot m\otimes\beta_i, \quad m\in{}_AM, \tag{3.6a}$$

$$M_{\hat{A}}: \qquad m\cdot\varphi := \langle m_{-1},\varphi\rangle m_0, \quad m\in{}^AM, \varphi\in\hat{A} \tag{3.6b}$$

and the invariants (coinvariants) with respect to A become coinvariants (invariants) with respect to $\hat{A}$.

In case when A is not only a WBA, but also a WHA one can say more about the invariants and coinvariants of (multiple) WHMs:

Lemma 3.1 *Let A be a WHA.*

i) The coinvariants and the invariants of a WHM of A can be equivalently characterized as

$$\begin{aligned} C(M_A^A) &= \{m\in M\,|\,\delta_R(m)=m\cdot\mathbf{1}^{(1)}\otimes\mathbf{1}^{(2)}\}, \\ C({}_AM^A) &= \{m\in M\,|\,\delta_R(m)=\mathbf{1}^{(1)}\cdot m\otimes\mathbf{1}^{(2)}\}, \\ C({}_A^AM) &= \{m\in M\,|\,\delta_L(m)=\mathbf{1}^{(1)}\otimes\mathbf{1}^{(2)}\cdot m\}, \\ C({}^AM_A) &= \{m\in M\,|\,\delta_L(m)=\mathbf{1}^{(1)}\otimes m\cdot\mathbf{1}^{(2)}\}, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} I(M_A^A) &= \{m\in M\,|\,m_0\cdot a\otimes m_1=m_0\otimes m_1S(a), a\in A\}, \\ I({}_AM^A) &= \{m\in M\,|\,m_0\otimes am_1=S(a)\cdot m_0\otimes m_1, a\in A\}, \\ I({}_A^AM) &= \{m\in M\,|\,m_{-1}\otimes a\cdot m_0=S(a)m_{-1}\otimes m_1, a\in A\}, \\ I({}^AM_A) &= \{m\in M\,|\,m_{-1}a\otimes m_0=m_{-1}\otimes m_0\cdot S(a), a\in A\}. \end{aligned} \tag{3.7b}$$

ii) The following maps define projections from WHMs onto their coinvariants and invariants, respectively

$$\begin{aligned} P^A(m) &:= m_0\cdot S(m_1),\ m\in M_A^A, &\quad \bar{P}^A(m) &:= S^{-1}(m_1)\cdot m_0,\ m\in{}_AM^A, \\ {}^AP(m) &:= S(m_{-1})\cdot m_0,\ m\in{}_A^AM, &\quad {}^A\bar{P}(m) &:= m_0\cdot S^{-1}(m_{-1}),\ m\in{}^AM_A, \end{aligned} \tag{3.8a}$$

$$P_A(m) := m_0 \cdot R(m_1),\ m \in M_A^A, \qquad {}_A\bar{P}(m) := \bar{L}(m_1) \cdot m_0,\ m \in {}_AM^A,$$
$${}_AP(m) := L(m_{-1}) \cdot m_0,\ m \in {}_A^AM, \qquad \bar{P}_A(m) := m_0 \cdot \bar{R}(m_{-1}),\ m \in {}^AM_A, \tag{3.8b}$$

where S is the antipode and $R, \bar{R}, L, \bar{L}$ are the projection maps (1.17) to integrals in the WHA A.

iii) In case of the multiple WHMs ${}_AM_A^A$ and ${}_A^AM_A$ the coinvariants are left and right A-modules with respect to the induced left and right adjoint actions, respectively.

Proof. i) The characterization (3.7a) of coinvariants and the form (3.8a) of the projections onto them have been already proved in [2]. Concerning the invariants of M_A^A first we note that the set given in (3.7b) is contained in the set of invariants defined in (3.5) since

$$\begin{aligned} m \cdot a &= (id \otimes \varepsilon)(m_0 \cdot a^{(1)} \otimes m_1 a^{(2)}) = (id \otimes \varepsilon)(m_0 \otimes m_1 S(a^{(1)}) a^{(2)}) \\ &= (id \otimes \varepsilon)(m_0 \otimes m_1 \Pi^R(a)) = (id \otimes \varepsilon)(\delta_R(m \cdot \Pi^R(a)) = m \cdot \Pi^R(a), \end{aligned} \tag{3.9}$$

for all $a \in A$. Using the third identity in (1.8) the opposite containment is as follows

$$\begin{aligned} m_0 \cdot a \otimes m_1 &= m_0 \cdot \mathbf{1}^{(1)} a \otimes m_1 \mathbf{1}^{(2)} = m_0 \cdot a^{(1)} \otimes m_1 \Pi^L(a^{(2)}) \\ &= m_0 \cdot a^{(1)} \otimes m_1 a^{(2)} S(a^{(3)}) = (m \cdot a^{(1)})_0 \otimes (m \cdot a^{(1)})_1 S(a^{(2)}) \\ &= (m \cdot \Pi^R(a^{(1)}))_0 \otimes (m \cdot \Pi^R(a^{(1)}))_1 S(a^{(2)}) = m_0 \otimes m_1 \Pi^R(a^{(1)}) S(a^{(2)}) \\ &= m_0 \otimes m_1 S(a), \qquad a \in A, m \in I(M_A^A). \end{aligned} \tag{3.10}$$

The cases of the other three WHMs can be proved similarly.

ii) The image of the map P_A is in $I(M_A^A)$ due to the defining property (1.9) of the right integrals in A. Applying P_A to an invariant $m \in I(M_A^A)$ and using their characterization (3.7b) and the non-degeneracy property (3.3)

$$\begin{aligned} m_0 \cdot R(m_1) &:= m_0 \cdot ((m_1 b_i) \leftharpoonup \hat{S}^2(\beta_i)) = m_0 \cdot ((m_1 S(b_i)) \leftharpoonup \hat{S}(\beta_i)) \\ &= (m_0 \cdot b_i) \cdot (m_1 \leftharpoonup \hat{S}(\beta_i)) = m_0 \cdot b_i m_1^{(2)} \langle m_1^{(1)}, \hat{S}(\beta_i) \rangle \\ &= m_0 \cdot S(m_1^{(1)}) m_1^{(2)} = m_0 \cdot \Pi^R(m_1) = m, \quad m \in I(M_A^A) \end{aligned} \tag{3.11}$$

follows, that is P_A is a projection onto the invariants of M_A^A. The cases of projections onto the invariants of the other three WHMs can be proved similarly.

iii) We have to show that the maps

$$\begin{aligned} \nu_L(a \otimes m) &\equiv a \star m := a^{(1)} \cdot m \cdot S(a^{(2)}), \qquad a \in A,\ m \in C({}_AM_A^A), \\ \nu_R(m \otimes a) &\equiv m \star a := S(a^{(1)}) \cdot m \cdot a^{(2)}, \qquad a \in A,\ m \in C({}_A^AM_A) \end{aligned} \tag{3.12}$$

provide a left and a right A-module structure $(C({}_AM_A^A), \nu_L)$ and $(C({}_A^AM_A), \nu_R)$, respectively. The image of the map ν_L is in $C({}_AM_A^A)$, because for all $a \in A$ and $m \in C({}_AM_A^A)$

$$\begin{aligned} \delta_R(a^{(1)} \cdot m \cdot S(a^{(2)})) &= a^{(11)} \cdot (\mathbf{1}^{(1)} \cdot m) \cdot S(a^{(2)})^{(1)} \otimes a^{(12)} \mathbf{1}^{(2)} S(a^{(2)})^{(2)} \\ &= a^{(1)} \cdot m \cdot S(a^{(4)}) \otimes a^{(2)} S(a^{(3)}) \\ &= a^{(1)} \cdot m \cdot S(a^{(3)}) \otimes \Pi^L(a^{(2)}) \in M \otimes A^L. \end{aligned} \tag{3.13}$$

The map ν_L is clearly a left A-action, i.e. $a \star (b \star m) = ab \star m$, for all $a, b \in A$ and $m \in C({}_AM_A^A)$, moreover, for all $m \in C({}_AM_A^A)$

$$\begin{aligned} m &= \mathbf{1} \cdot m = (\mathbf{1} \cdot m)_0 \cdot \Pi^R((\mathbf{1} \cdot m)_1) = \mathbf{1}^{(1)} \cdot m_0 \cdot \Pi^R(\mathbf{1}^{(2)} m_1) \\ &= \mathbf{1}^{(1)} \cdot m \cdot \Pi^R(\mathbf{1}^{(2)}) = \mathbf{1}^{(1)} \cdot m \cdot S(\mathbf{1}^{(2)}) = \mathbf{1} \star m, \end{aligned} \tag{3.14}$$

where we used the identities (3.3) and (1.11) and the property (3.7a) of the coinvariants. The proof of the case $(C({}_A^AM_A), \nu_R)$ is similar. ■

Extending the result of [2] concerning the structure of a WHM the structure of a multiple WHM is given by the following

Theorem 3.2 *i) Let ${}_AM_A^A$ be a multiple weak Hopf module of the WHA A. Then ${}_AM_A^A$ is isomorphic as a multiple WHM to ${}_A(C(M) \times A_A^A)$, which as a left A-module is isomorphic to the product of the left A-modules $(C(M), \star)$ defined in the previous Lemma and the left regular module ${}_AA$, while the right A-module and right A-comodule structures are inherited from the WHM $A_A^A \equiv (A, \mu, \Delta)$.*

ii) In particular, ${}_A\hat{A}_A^A \equiv (\hat{A}, \mu_L, \mu_R, \delta_R)$ is a multiple WHM with structure maps

$$\mu_L(a \otimes \varphi) \equiv a \cdot \varphi := \varphi \leftharpoonup S^{-1}(a), \tag{3.15a}$$

$$\mu_R(\varphi \otimes a) \equiv \varphi \cdot a := S(a) \rightharpoonup \varphi, \qquad a \in A, \varphi \in \hat{A}, \tag{3.15b}$$

$$\delta_R(\varphi) \equiv \varphi_0 \otimes \varphi_1 := \beta_i \varphi \otimes b_i, \tag{3.15c}$$

where $\{b_i\} \subset A$ and $\{\beta_i\} \subset \hat{A}$ are dual bases with respect to the canonical pairing, therefore

$${}_A\hat{A}_A^A \simeq {}_A(C(\hat{A}) \times A_A^A) = {}_A(\hat{I}^L \times A_A^A), \tag{3.16}$$

where $\hat{I}^L$ is the space of left integrals in the WHA $\hat{A}$.

Proof. i) As a k-linear space ${}_A(C(M) \times A_A^A) \equiv (C(M) \times A, \mu_L, \mu_R, \delta_R)$ is (see (2.2a))

$$\begin{aligned} C(M) \times A &:= \mathbf{1}^{(1)} \star C(M) \otimes \mathbf{1}^{(2)} A = \mathbf{1}^{(1)} \cdot C(M) \cdot S(\mathbf{1}^{(2)}) \otimes \mathbf{1}^{(3)} A \\ &= C(M) \cdot \mathbf{1}^{(1)} S(\mathbf{1}^{(2)}) \otimes \mathbf{1}^{(3)} A = C(M) \cdot \Pi^L(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} A \\ &= C(M) \cdot S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} A \end{aligned} \tag{3.17}$$

due to the fact that $x^R \cdot m = x^R \cdot (\mathbf{1} \star m) := x^R \mathbf{1}^{(1)} \cdot m \cdot S(\mathbf{1}^{(2)}) = m \cdot x^R; m \in C(M), x^R \in A^R$, which follows from the identities (3.14) and (1.12). One can easily check that the maps

$$\begin{aligned} a \cdot (\sum_i n_i \otimes b_i) &:= \sum_i a^{(1)} \star n_i \otimes a^{(2)} b_i, \\ (\sum_i n_i \otimes b_i) \cdot a &:= \sum_i n_i \otimes b_i a, \qquad\qquad a \in A,\ \sum_i n_i \otimes b_i \in C(M) \times A. \\ \delta_R(\sum_i n_i \otimes b_i) &:= \sum_i n_i \otimes b_i^{(1)} \otimes b_i^{(2)}, \end{aligned} \tag{3.18}$$

provide $C(M)\times A$ with a multiple WHM-structure. The k-linear maps $U\colon {}_AM_A^A \to C(M)\times A$ and $V\colon C(M)\times A \to {}_AM_A^A$ defined by

$$U(m) := m_0 \cdot S(m_1)\otimes m_2, \qquad V(\sum_i n_i\otimes b_i) := \sum_i n_i\cdot b_i \tag{3.19}$$

are inverses of each other [2], i.e. $V\circ U = id_{{}_AM_A^A}$ and $U\circ V = id_{C(M)\times A}$. In order to prove that ${}_AM_A^A$ and ${}_A(C(M)\times A_A^A)$ are isomorphic as multiple WHMs as well, we have to show that both U and V are left and right A-module and right A-comodule maps. We can restrict ourselves to the left A-module properties, because the two other properties were already shown in [2]:

$$\begin{aligned} U(a\cdot m) &= (a\cdot m)_0\cdot S((a\cdot m)_1)\otimes (a\cdot m)_2 \\ &= a^{(1)}\cdot m_0\cdot S(m_1)S(a^{(2)})\otimes a^{(3)}m_2 \\ &= a^{(1)}\star (m_0\cdot S(m_1))\otimes a^{(2)}m_2 \\ &= a\cdot U(m), \quad a\in A,\, m\in M, \end{aligned} \tag{3.20a}$$

$$\begin{aligned} V(a\cdot(\sum_i n_i\otimes b_i)) &= \sum_i V(a^{(1)}\star n_i\otimes a^{(2)}b_i) = \sum_i V(a^{(1)}\cdot n_i\cdot S(a^{(2)})\otimes a^{(3)}b_i) \\ &= \sum_i a^{(1)}\cdot n_i\cdot S(a^{(2)})a^{(3)}b_i = \sum_i a^{(1)}\cdot n_i\cdot \Pi^R(a^{(2)})b_i \\ &= \sum_i a^{(1)}\Pi^R(a^{(2)})\cdot n_i\cdot b_i = a\cdot(\sum_i n_i\cdot b_i) \\ &= a\cdot V(\sum_i n_i\otimes b_i), \quad a\in A,\ \sum_i n_i\otimes b_i\in C(M)\times A. \end{aligned} \tag{3.20b}$$

ii) The WHM structure $\hat{A}_A^A \equiv (\hat{A},\mu_R,\delta_R)$ given by (3.15b–c) of the multiple WHM ${}_A\hat{A}_A^A$ has been shown in [2]. The map μ_L defined in (3.15a) is clearly a left A-module map on $\hat{A}$ that commutes with the given right A-module map μ_R. The right comodule map δ_R is also a left A-module map, because

$$\begin{aligned} \delta_R(a\cdot\varphi) &:= \delta_R(\varphi \leftharpoondown S^{-1}(a)) = \delta_R(\varphi\leftharpoondown \bar{\Pi}^L(a^{(2)})S^{-1}(a^{(1)})) \\ &= \delta_R(((\hat{\mathbf{1}}\leftharpoondown \bar{\Pi}^L(a^{(2)}))\varphi)\leftharpoondown S^{-1}(a^{(1)})) = \delta_R(((\hat{\mathbf{1}}\leftharpoondown a^{(2)})\varphi)\leftharpoondown S^{-1}(a^{(1)})) \\ &= \delta_R(((\hat{\mathbf{1}}\leftharpoondown a^{(3)}S^{-1}(a^{(2)}))(\varphi\leftharpoondown S^{-1}(a^{(1)})) \\ &= \beta_i(\hat{\mathbf{1}}\leftharpoondown \bar{\Pi}^R(a^{(2)}))(\varphi\leftharpoondown S^{-1}(a^{(1)}))\otimes b_i \\ &= (\beta_i\leftharpoondown \bar{\Pi}^R(a^{(2)}))(\varphi\leftharpoondown S^{-1}(a^{(1)}))\otimes b_i = ((\beta_i\leftharpoondown a^{(2)})\varphi)\leftharpoondown S^{-1}(a^{(1)})\otimes b_i \\ &= (\beta_i\varphi)\leftharpoondown S^{-1}(a^{(1)})\otimes a^{(2)}b_i = a^{(1)}\cdot\varphi_0\otimes a^{(2)}\varphi_1, \qquad a\in A,\varphi\in\hat{A}, \end{aligned} \tag{3.21}$$

where we used the identities (1.6) and (1.10–11). Hence, the maps (3.15) provides $\hat{A}$ with a multiple WHM structure, and the statement (3.16) follows from the previously proved structure of a general multiple weak Hopf module. By dualizing the right A-coaction to left $\hat{A}$-action as in (3.6) the right coinvariants $C(\hat{A}^A)$ become the left invariants of the left regular module ${}_{\hat{A}}\hat{A}$, which is the space of left integrals $\hat{I}^L$ in $\hat{A}$. ■

Remark 3.3 The k-dual $\hat{A}_A := (\hat{A}, \leftharpoonup)$ of the left regular A-module ${}_AA$ is projective, because the inverse of the antipode provides the isomorphism of the right A-modules

$$\hat{S}^{-1}\colon \hat{A}_A \to (\hat{A}, \mu_R := \rightharpoonup \circ S)$$

and the structure theorem of weak Hopf modules implies that the latter is isomorphic to a direct summand of the free right A-module $\hat{I}^L \otimes A_A$. Projectivity of $\hat{A}_A$ implies the injectivity of its k-dual, that is of ${}_AA$. Hence, A is a quasi-Frobenius algebra [5], which has been already established in [2]. ■

Corollary 3.4 *${}_AI^R$ and A^R_A are A-duals of each other. ${}_A\hat{I}^L$ is the right conjugate module ${}_A\overrightarrow{I^R}$ of ${}_AI^R$ and they are direct sum of irreducible submodules*

$$ {}_AI^R = \bigoplus_p {}_AI^R_p, \qquad I^R_p := z^R_p I^R, \tag{3.22a}$$

$$ {}_A\hat{I}^L = \bigoplus_p {}_A\hat{I}^L_p, \qquad \hat{I}^L_p := z^R_p \star \hat{I}^L, \tag{3.22b}$$

where $\{z^R_p\}_p$ is the complete set of primitive idempotents in $Z^R := A^R \cap \mathrm{Center}\, A$.

Proof. Since the right integrals I^R form a left ideal in A and ${}_AA$ is injective it follows [5] that every $\phi \in \mathrm{Hom}\,({}_AI^R, {}_AA)$ can be extended to $\bar{\phi} \in \mathrm{Hom}\,({}_AA, {}_AA)$. But every such element $\bar{\phi}$ is given by a right multiplication of an element $a \in A$, hence any ϕ is given by the restriction of a right multiplication to I^R:

$$\phi(r) = \bar{\phi}(r) \equiv ra = r\Pi^R(a), \quad r \in I^R. \tag{3.23}$$

This establish that $\mathrm{Hom}\,({}_AI^R, {}_AA) \simeq A^R$ as a k-linear space. The identical right module structure, $({}_AI^R)'_A := \mathrm{Hom}\,({}_AI^R, {}_AA)_A \simeq A^R_A$, follows from

$$\begin{aligned}(\phi_x \cdot a)(r) &:= \phi_x(r)a = (rx)a = r\Pi^R(xa) =: r(x \cdot a)\\ &= \phi_{x\cdot a}(r), \quad x \in A^R, r \in I^R, a \in A.\end{aligned} \tag{3.24}$$

For the other duality relation one first notices that elements $f \in \mathrm{Hom}\,(A^R_A, A_A)$ can be characterized by left multiplication with the image $f(\mathbf{1}) \in A$,

$$f(x) = f(\mathbf{1} \cdot x) = f(\mathbf{1})x, \quad x \in A^R, \tag{3.25}$$

which should be a right integral, $f(\mathbf{1}) \in I^R$, because the module homomorphism property requires

$$f(\mathbf{1})a = f(\mathbf{1} \cdot a) := f(\Pi^R(\mathbf{1}a)) = f(\mathbf{1})\Pi^R(a). \tag{3.26}$$

The common left A-module strucure is immediate since it is given by left multiplication on the image $f(\mathbf{1}) \in I^R$ in both cases.

Since in quasi-Frobenius algebras the A-duals of irreducible right A-modules are irreducible left A-modules the direct sum decomposition (3.22a) into irreducible submodules is induced by the corresponding decomposition (2.25) of A^R_A.

$\overrightarrow{{}_AI^R} = {}_A\hat{I}^L$ follows, because the left A-module structures of ${}_A\hat{I}^L$ and ${}_AI^R$ are related as it is required in (2.12)

$$\begin{aligned}\langle a\star\lambda, r\rangle &:= \langle a^{(1)}\cdot\lambda\cdot S(a^{(2)}), r\rangle = \langle S^2(a^{(2)})\rightharpoonup\lambda\leftharpoonup S^{-1}(a^{(1)}), r\rangle\\ &= \langle\lambda, S^{-1}(a^{(1)})rS^2(a^{(2)})\rangle = \langle\lambda, S^{-1}(a^{(1)})r\Pi^R(S^2(a^{(2)}))\rangle\\ &= \langle\lambda, S^{-1}(a^{(1)})rS^2(\Pi^R(a^{(2)}))\rangle = \langle\lambda, S^{-1}(a^{(1)})rS^3(\bar\Pi^L(a^{(2)}))\rangle\\ &= \langle\lambda, S^{-1}(\mathbf{1}^{(1)})S^{-1}(a)rS^3(\mathbf{1}^{(2)})\rangle = \langle\mathbf{1}\star\lambda, S^{-1}(a)r\rangle\\ &= \langle\lambda, S^{-1}(a)r\rangle,\quad a\in A, \lambda\in\hat{I}^L, r\in I^R\end{aligned}\tag{3.27}$$

and the restriction of the canonical pairing to these integrals is non-degenerate. Hence, ${}_A\hat{I}^L$ is also semisimple and the decomposition (3.22b) follows because z^R_p is a central projection in A and $I^R_p := z^R_pI^R = I^Rz^R_p = I^RS^{-1}(z^R_p) = S^{-1}(z^R_p)I^R$. ∎

4. Existence of non-degenerate left integrals in weak Hopf algebras

Here we prove the generalization of the Larson–Sweedler [10] theorem for weak Hopf algebras.

Theorem 4.1 *A finite dimensional weak bialgebra A over a field k is a weak Hopf algebra iff there exists a non-degenerate left integral in A.*

Proof. Sufficiency. A left integral $l\in A$ obeys the defining property $al = \Pi^L(a)l, a\in A$. Non-degeneracy means that the maps

$$\begin{aligned}R_l\colon \hat{A} &\to A & L_l\colon \hat{A} &\to A\\ \varphi &\mapsto (\varphi\rightharpoonup l) & \varphi &\mapsto (l\leftharpoonup\varphi)\end{aligned}$$

are bijections. This implies that there exist $\lambda,\rho\in\hat{A}$ such that $l\leftharpoonup\rho\equiv L_l(\rho) = \mathbf{1} = R_l(\lambda)\equiv\lambda\rightharpoonup l$. Let us define the k-linear maps $S\colon A\to A$ and $\hat{S}\colon\hat{A}\to\hat{A}$ by

$$\begin{aligned}S(a) &:= (R_l\circ\hat{L}_\lambda)(a)\equiv(\lambda\leftharpoonup a)\rightharpoonup l = l^{(1)}\langle al^{(2)},\lambda\rangle,\\ \hat{S}(\psi) &:= (\hat{R}_\lambda\circ L_l)(\psi)\equiv(l\leftharpoonup\psi)\rightharpoonup\lambda = \lambda^{(1)}\langle\psi\lambda^{(2)}, l\rangle.\end{aligned}\tag{4.1}$$

They are transposed to each other with respect to the canonical pairing and $\hat{S}(\rho) = \lambda$. Now we prove that λ (ρ) is a nondegenerate left (right) integral in $\hat{A}$ obeying $l\rightharpoonup\lambda = \hat{\mathbf{1}} = l\rightharpoonup\rho$.

Since R_l and L_l are bijections the identities

$$\begin{aligned}R_l(\psi\lambda) &= (\psi\lambda)\rightharpoonup l = \psi\rightharpoonup(\lambda\rightharpoonup l) = \psi\rightharpoonup\mathbf{1} = \hat\Pi^L(\psi)\rightharpoonup\mathbf{1} = \hat\Pi^L(\psi)\rightharpoonup(\lambda\rightharpoonup l)\\ &= R_l(\hat\Pi^L(\psi)\lambda)\\ L_l(\rho\psi) &= l\leftharpoonup(\rho\psi) = (l\leftharpoonup\rho)\leftharpoonup\psi = \mathbf{1}\leftharpoonup\psi = \mathbf{1}\leftharpoonup\hat\Pi^R(\psi) = (l\leftharpoonup\rho)\leftharpoonup\hat\Pi^R(\psi)\\ &= L_l(\rho\hat\Pi^R(\psi))\end{aligned}\tag{4.2}$$

imply that λ (ρ) is a left (right) integral in $\hat{A}$. Using the properties $l \leftharpoonup \rho = \mathbf{1} = \lambda \rightharpoonup l$

$$\begin{aligned} \hat{\Pi}^R(l \rightharpoonup \rho) &= \hat{\Pi}^R(\rho^{(1)})\langle \rho^{(2)}, l\rangle = \hat{\mathbf{1}}^{(1)}\langle \rho\hat{\mathbf{1}}^{(2)}, l\rangle = \hat{\mathbf{1}}^{(1)}\langle \hat{\mathbf{1}}^{(2)}, l \leftharpoonup \rho\rangle = \hat{\mathbf{1}}, \\ \hat{\bar{\Pi}}^R(l \rightharpoonup \lambda) &= \hat{\bar{\Pi}}^R(\lambda^{(1)})\langle \lambda^{(2)}, l\rangle = \hat{\mathbf{1}}^{(1)}\langle \hat{\mathbf{1}}^{(2)}\lambda, l\rangle = \hat{\mathbf{1}}^{(1)}\langle \hat{\mathbf{1}}^{(2)}, \lambda \rightharpoonup l\rangle = \hat{\mathbf{1}}, \end{aligned} \tag{4.3}$$

which imply that $l \rightharpoonup \rho = \hat{\mathbf{1}} = l \rightharpoonup \lambda$ since $l \rightharpoonup \rho, l \rightharpoonup \lambda \in \hat{A}^L$ and both of the $\hat{A}^L - A^R$ and $\hat{A}^L - A^L$ pairings are nondegenerate. The proved properties of $\lambda, \rho \in \hat{A}$ allow us to construct the inverse of the map $\hat{S}$:

$$\hat{S}^{-1}(\psi) := (\hat{R}_\rho \circ R_l)(\psi) \equiv (\psi \rightharpoonup l) \rightharpoonup \rho = \rho^{(1)}\langle \rho^{(2)}\psi, l\rangle. \tag{4.4}$$

Indeed, for all $\psi \in \hat{A}$ one obtains

$$\begin{aligned} (\hat{S}^{-1} \circ \hat{S})(\psi) &:= \rho^{(1)}\langle \rho^{(2)}\lambda^{(1)}, l\rangle\langle \psi\lambda^{(2)}, l\rangle = \rho^{(1)}\langle \rho^{(2)}\hat{\Pi}^R(\psi^{(1)})\lambda^{(1)}, l\rangle\langle \psi^{(2)}\lambda^{(2)}, l\rangle \\ &= \rho^{(1)}\psi^{(1)}\langle \rho^{(2)}\psi^{(2)}\lambda^{(1)}, l\rangle\langle \psi^{(3)}\lambda^{(2)}, l\rangle \\ &= \rho^{(1)}\psi^{(1)}\langle \rho^{(2)}\hat{\Pi}^L(\psi^{(2)})\lambda^{(1)}, l\rangle\langle \lambda^{(2)}, l\rangle \\ &= \rho^{(1)}\psi^{(1)}\langle \rho^{(2)}\hat{\Pi}^L(\psi^{(2)}), l\rangle = \rho^{(1)}\psi\langle \rho^{(2)}, l\rangle = \psi. \end{aligned} \tag{4.5}$$

Therefore the transposed map $S^{-1} := (\hat{S}^{-1})^t \equiv L_l \circ \hat{L}_\rho$ is the inverse of S and invertibility of $S := R_l \circ \hat{L}_\lambda$ given in (4.1) implies that $\hat{L}_\lambda$ and $\hat{L}_\rho$ are invertible, hence λ and ρ are nondegenerate left and right integrals in $\hat{A}$, respectively.

Since the right integral $\rho \in \hat{A}$ is nondegenerate there exists $r \in A$ such that $\rho \leftharpoonup r = \hat{\mathbf{1}}$. In a similar way as before one proves that r is a right integral obeying $r \leftharpoonup \rho = \mathbf{1}$:

$$\begin{aligned} \hat{L}_\rho(ra) &= \rho \leftharpoonup (ra) = \hat{\mathbf{1}} \leftharpoonup a = \hat{\mathbf{1}} \leftharpoonup \Pi^R(a) = \hat{L}_\rho(r\Pi^R(a)) \\ \bar{\Pi}^L(r \leftharpoonup \rho) &= \langle \rho, r^{(1)}\rangle\bar{\Pi}^L(r^{(2)}) = \langle \rho, r\mathbf{1}^{(1)}\rangle\mathbf{1}^{(2)} = \langle \rho \leftharpoonup r, \mathbf{1}^{(1)}\rangle\mathbf{1}^{(2)} = \mathbf{1}, \end{aligned} \tag{4.6}$$

hence $r \leftharpoonup \rho = \mathbf{1}$ follows since $r \leftharpoonup \rho \in A^R$ and the $A^R - \hat{A}^R$ pairing is nondegenerate. But then $S = L_r \circ \hat{R}_\rho$ also holds (therefore r is nondegenerate), because

$$\begin{aligned} (S^{-1} \circ L_r \circ \hat{R}_\rho)(a) &= (L_l \circ \hat{L}_\rho \circ L_r \circ \hat{R}_\rho)(a) = \langle r^{(1)}a, \rho\rangle\langle r^{(2)}l^{(1)}, \rho\rangle l^{(2)} \\ &= \langle r^{(1)}a^{(1)}, \rho\rangle\langle r^{(2)}\Pi^L(a^{(2)})l^{(1)}, \rho\rangle l^{(2)} = \langle r^{(1)}a^{(1)}, \rho\rangle\langle r^{(2)}a^{(2)}l^{(1)}, \rho\rangle a^{(3)}l^{(2)} \\ &= \langle r^{(1)}, \rho\rangle\langle r^{(2)}\Pi^R(a^{(1)})l^{(1)}, \rho\rangle a^{(2)}l^{(2)} = \langle \Pi^R(a^{(1)})l^{(1)}, \rho\rangle a^{(2)}l^{(2)} \\ &= \langle l^{(1)}, \rho\rangle al^{(2)} = a. \end{aligned} \tag{4.7}$$

Now, the three defining properties (1.2a–c) of the antipode fulfill for the map $S := R_l \circ \hat{L}_\lambda = L_r \circ \hat{R}_\rho$, because for all $a \in A$ one has

$$a^{(1)}S(a^{(2)}) = a^{(1)}l^{(1)}\langle a^{(2)}l^{(2)}, \lambda\rangle = \Pi^L(a)l^{(1)}\langle l^{(2)}, \lambda\rangle = \Pi^L(a), \tag{4.8a}$$

$$S(a^{(1)})a^{(2)} = r^{(2)}a^{(2)}\langle r^{(1)}a^{(1)}, \rho\rangle = r^{(2)}\Pi^R(a)\langle r^{(1)}, \rho\rangle = \Pi^R(a), \tag{4.8b}$$

$$\begin{aligned} S(a^{(1)})a^{(2)}S(a^{(3)}) &= \Pi^R(a^{(1)})S(a^{(2)}) = \Pi^R(a^{(1)})l^{(1)}\langle a^{(2)}l^{(2)}, \lambda\rangle \\ &= l^{(1)}\langle al^{(2)}, \lambda\rangle = S(a). \end{aligned} \tag{4.8c}$$

Necessity. The statement follows from Lemma 2.6 and results (Theorem 3.16) in [2], but for completeness we give a full proof.

Applying the structure theorem of multiple WHMs to ${}_A\hat{A}^A_A$ given by (3.15) we get the isomorphism ${}_A\hat{A}^A_A \simeq {}_A(\hat{I}^L \times A^A_A)$. Therefore if we can prove that the restriction of the left A-module structure of ${}_A\hat{I}^L \equiv ({}_A\hat{I}^L, \star)$ (see Lemma 3.1 iii)) to $A^R \subset A$ leads to a free A^R-module ${}_{A^R}\hat{I}^L \equiv ({}_{A^R}\hat{I}^L, \star)$ with a single generator $\lambda_0 \in \hat{I}^L$ we are ready, because using the multiple WHM isomorphism $V\colon \hat{I}^L \times A \to \hat{A}$ given in (3.19) and the presence of the separating idempotent in $\hat{I}^L \times A := \mathbf{1}^{(1)} \star \hat{I}^L \otimes \mathbf{1}^{(2)} \cdot A = \hat{I}^L \cdot S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} A$

$$\begin{aligned}\hat{A} &= V(\hat{I}^L \times A) = V((A^R \star \lambda_0) \times A) = V((\lambda_0 \cdot S(A^R)) \times A) = V(\lambda_0 \times S(A^R)A) \\ &= V(\lambda_0 \times A) := \lambda_0 \cdot A := S(A) \rightharpoonup \lambda_0 = A \rightharpoonup \lambda_0\end{aligned} \tag{4.9}$$

implies the non-degeneracy of the left integral λ_0 in the weak Hopf algebra $\hat{A}$.

To prove this assertion it is enough to show that there exists a permutation σ of primitive central idempotents $\{e^R_i\}_i \subset A^R$ such that $\mathrm{End}_{A^R}(e^R_i \star \hat{I}^L) \simeq S(e^R_{\sigma(i)})A^L$ as algebras and it is given by $S(e^R_{\sigma(i)})A^L\star$, that is by the restriction of the left A-action $\star$ on $\hat{I}^L$ to $A^L \subset A$. Indeed, $\hat{I}^L$ is faithful as a left A^R- or left A^L-module due to Corollary 3.4 and Lemma 2.6 ii), therefore the assumptions $A^L \simeq \mathrm{End}_{A^R}\hat{I}^L = A^L\star$ and the existence of the permutation σ imply that as a left A^T-module $({}_{A^T}\hat{I}^L, \star)$ decomposes into direct sum of irreducible A^T-modules:

$$\hat{I}^L = \bigoplus_i \hat{I}^L_{\sigma(i)i}, \qquad \hat{I}^L_{\sigma(i)i} := e^L_{\sigma(i)} e^R_i \star \hat{I}^L, \quad |\hat{I}^L_{\sigma(i)i}| = |D^R_i| n_{\sigma(i)} n_i, \tag{4.10}$$

where $\{e^R_i\}_i \subset A^R$ and $\{e^L_i := S(e^R_i)\}_i \subset A^L$ are complete sets of primitive central idempotents, $D^{R/L}_i$ is the division algebra corresponding to the Wedderburn component $e^{R/L}_i A^{R/L} \simeq M_{n_i}(D^{R/L}_i)$ of the separable algebra $A^{R/L}$ obeying $D^L_{\sigma(i)} \simeq S(D^R_i)$. But due to Lemma 2.6 i) we have the following inequality for the k-dimensions

$$\sum_i |D^R_i| n_{\sigma(i)} n_i = |\hat{I}^L| \geq |A^R| = \sum_i |D^R_i| n^2_i, \tag{4.11}$$

which can be fulfilled only if $n_{\sigma(i)} = n_i$ due to the Cauchy–Schwartz inequality, which leads to the opposite estimate. However, in the case $n_{\sigma(i)} = n_i$ the left A^R-module $({}_{A^R}\hat{I}^L, \star)$ and the left A^L-module $({}_{A^L}\hat{I}^L, \star)$ are isomorphic to the left regular A^R-module ${}_{A^R}A^R$ and the left regular A^L-module ${}_{A^L}A^L$, that is $({}_{A^{R/L}}\hat{I}^L, \star)$ becomes a free left $A^{R/L}$-module with a single generator by restricting the A-action to these subalgebras.

In order to show $A^L \simeq \mathrm{End}_{A^R}\hat{I}^L = A^L\star$, first we note that $\mathrm{End}_k\hat{I}^L$ can be realized by $\hat{I}^L \otimes I^R$ as $(\sum_\alpha \lambda_\alpha \otimes r_\alpha)(\lambda) := \sum_\alpha \lambda_\alpha \langle r_\alpha, \lambda\rangle, \lambda \in \hat{I}^L$, due to the non-degeneracy of the restriction of the canonical pairing to these integrals. The subalgebra $\mathrm{End}_{A^R}\hat{I}^L \subset \mathrm{End}_k\hat{I}^L$ is given by $\hat{I}^L \times I^R$. Indeed, if $f = \sum_\alpha \lambda_\alpha \otimes r_\alpha \in \mathrm{End}_{A^R}\hat{I}^L$ then using (3.27) we get

$$\begin{aligned}f(\lambda) &= f(\mathbf{1} \star \lambda) = f(\mathbf{1}^{(1)} S(\mathbf{1}^{(2)}) \star \lambda) = \mathbf{1}^{(1)} \star f(S(\mathbf{1}^{(2)}) \star \lambda) \\ &= \sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \langle r_\alpha, S(\mathbf{1}^{(2)}) \star \lambda\rangle = \sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \langle \mathbf{1}^{(2)} r_\alpha, \lambda\rangle \\ &= (\sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \otimes \mathbf{1}^{(2)} r_\alpha)(\lambda), \quad \lambda \in \hat{I}^L,\end{aligned} \tag{4.12}$$

that is End $_{A^R}\hat{I}^L \subset \hat{I}^L \times I^R$. Choosing $f = \sum_\alpha \lambda_\alpha \otimes r_\alpha \in \hat{I}^L \times I^R$ and $x^R \in A^R$

$$
\begin{aligned}
f(x^R \star \lambda) &\equiv (\sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \otimes \mathbf{1}^{(2)} r_\alpha)(x^R \star \lambda) = \sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \langle \mathbf{1}^{(2)} r_\alpha, x^R \star \lambda\rangle \\
&= \sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \langle S^{-1}(x^R)\mathbf{1}^{(2)} r_\alpha, \lambda\rangle = \sum_\alpha \mathbf{1}^{(1)} \star \lambda_\alpha \langle S^{-1}(S(\mathbf{1}^{(2)})x^R) r_\alpha, \lambda\rangle \\
&= \sum_\alpha x^R \mathbf{1}^{(1)} \star \lambda_\alpha \langle \mathbf{1}^{(2)} r_\alpha, \lambda\rangle = x^R \star (\sum_\alpha \lambda_\alpha \otimes r_\alpha)(\lambda), \qquad x^R \in A^R,\ \lambda \in \hat{I}^L,
\end{aligned}
\tag{4.13}
$$

leads to the opposite containment, hence End $_{A^R}\hat{I}^L = \hat{I}^L \times I^R$.

The structure of the subalgebra $\hat{I}^L \times I^R$ of $\hat{I}^L \otimes I^R$ can be obtained as follows. The restriction of the multiple WHM isomorphism $_A\hat{A}^A_A \simeq {}_A(\hat{I}^L \times A^A_A)$ to the right invariants leads to the isomorphism $_AI(\hat{A}_A) \simeq {}_AI(\hat{I}^L \times A_A)$ of left A-modules, where the left A-module structure of the right invariants is inherited from that of the corresponding multiple WHM. In our case $I(\hat{A}_A) = \hat{A}^L$ and $I(\hat{I}^L \times A_A) = \hat{I}^L \times I^R$. Indeed, the latter equality can be seen by using the form (3.8b) of the projection P_A to right invariants of the WHM $\hat{I}^L \times A^A_A$. To prove the former equality we note that $\hat{A}^L \subset I(\hat{A}_A)$ is the consequence of

$$
\begin{aligned}
\varphi^L \cdot a := S(a) \rightharpoonup \varphi^L &= \Pi^L(S(a)) \rightharpoonup \varphi^L = S(\Pi^R(a)) \rightharpoonup \varphi^L \\
&= \varphi^L \cdot \Pi^R(a), \qquad \varphi^L \in \hat{A}^L, a \in A,
\end{aligned}
\tag{4.14}
$$

while the opposite containment follows from

$$
\begin{aligned}
\langle \varphi, S(a)\rangle &= \hat{\varepsilon}(\varphi^{(1)})\langle \varphi^{(2)}, S(a)\rangle = \hat{\varepsilon}(\varphi \cdot a) = \hat{\varepsilon}(\varphi \cdot \Pi^R(a)) \\
&= \hat{\varepsilon}(\varphi^{(1)})\langle \varphi^{(2)}, S(\Pi^R(a))\rangle = \hat{\varepsilon}(\varphi^{(1)})\langle \varphi^{(2)}, \Pi^L(S(a))\rangle \\
&= \hat{\varepsilon}(\varphi^{(1)})\langle \hat{\Pi}^L(\varphi^{(2)}), S(a)\rangle = \langle \hat{\Pi}^L(\varphi), S(a)\rangle, \quad a \in A, \varphi \in I(\hat{A}_A),
\end{aligned}
\tag{4.15}
$$

using the identities (1.7) and (1.11). The restricted isomorphism $U\colon {}_A\hat{A}^L \to {}_A(\hat{I}^L \times I^R)$ with U given in (3.19) leads to

$$
U(\hat{\mathbf{1}}) = \hat{\mathbf{1}}_{00} \cdot S(\hat{\mathbf{1}}_{01}) \otimes \hat{\mathbf{1}}_1 = \sum_{i,j} S^2(b_j) \rightharpoonup (\beta_j \beta_i \hat{\mathbf{1}}) \otimes b_i = \sum_i \hat{L}(\beta_i) \otimes b_i = \sum_a \lambda_a \otimes r_a. \tag{4.16}
$$

Here $\{\lambda_a\}_a \subset \hat{I}^L$ and $\{r_a\}_a \subset I^R$ are dual bases with respect to the restriction of the canonical pairing and in the last equality of (4.16) we used the property (1.18) of the projection $\hat{L}$ defined in (1.17) onto left integrals in $\hat{A}$. Therefore

$$
\begin{aligned}
\text{End}\,_{A^R}\hat{I}^L &= \hat{I}^L \times I^R = U(\hat{A}^L) = U(A^L \cdot \hat{\mathbf{1}}) = A^L \cdot U(\hat{\mathbf{1}}) = A^L \cdot (\sum_a \lambda_a \otimes r_a) \\
&= \sum_a A^L \mathbf{1}^{(1)} \star \lambda_a \cdot \otimes \mathbf{1}^{(2)} r_a = \sum_a A^L \star \lambda_a \otimes r_a
\end{aligned}
\tag{4.17}
$$

leads to $A^L \simeq$ End $_{A^R}\hat{I}^L = A^L\star$, because $\sum_a \lambda_a \otimes r_a$ is just the unit element of End $_{A^R}\hat{I}^L$.

The property $A^L \simeq \mathrm{End}_{A^R}\hat{I}^L = A^L\star$ ensures that $\hat{I}^L$ decomposes into simple left A^T-submodules of $e_i^L e_j^R \star \hat{I}^L$ type, where $e_i^{L/R} \in A^{L/R}$ are primitive central idempotents. For the existence of the permutation σ required in (4.10) we have to show that the multiplicities $\hat{m}_{ij}$ of the simple left A^T-submodules $e_i^L e_j^R \star \hat{I}^L$ give rise to a permutation matrix $\hat{m}$. Since ${}_A\hat{I}^L$ is the right conjugate of the module ${}_AI^R$ (see Corollary 3.4) the module ${}_AI^R$ also becomes a direct sum of simple left A^T-submodules of $e_i^L e_j^R I^R$ type with 'transposed' multiplicities $m_{ij} = \hat{m}_{ji}$ by restricting the left A-action to the subalgebra A^T. Moreover, the above mentioned left A-module isomorphism $U\colon {}_A\hat{A}^L \to {}_A(\hat{I}^L \times I^R)$ relates the A^T-multiplicities $m^0_{ij} = \delta_{ij}$ of the module ${}_A\hat{A}^L$ to the A^T-multiplicities $\hat{m}_{ij}, m_{ij}$ of the integrals:

$$\delta_{ij} = m^0_{ij} = \sum_k \hat{m}_{ik} m_{kj}. \tag{4.18}$$

Hence, using the property $m_{kj} = \hat{m}_{jk}$ (4.18) implies that the integer valued multiplicity matrix $\hat{m}$ is orthogonal, i.e. it is a permutation matrix. ■

Corollary 4.2 A weak Hopf algebra A is a Frobenius algebra since a non-degenerate left integral in the dual weak Hopf algebra $\hat{A}$ provides a non-degenerate associative bilinear form on A. ■

5. Grouplike elements and invertible modules

Here we describe (left/right) grouplike elements in a WHA A and their role in the representation category of the dual WHA $\hat{A}$.

The set of *grouplike elements* $G(H)$ in a Hopf algebra H can be defined to be [17] $G(H) := \{g \in H | \Delta(g) = g \otimes g, \varepsilon(g) \neq 0\}$. The grouplike elements are linearly independent, they obey the property $S(g)g = \mathbf{1}$ and they form a group. The generalization of this notion to a weak Hopf algebra A

$$G(A) := \{g \in A | \Delta(\mathbf{1})(g \otimes g) = \Delta(g) = (g \otimes g)\Delta(\mathbf{1}), gS(g) = \mathbf{1}\}$$

given in [2] seems to be too restrictive, hence we introduce slightly softened generalizations as well:

Definition 5.1 *The set of right/left grouplike elements $G_{R/L}(A)$ in a weak Hopf algebra A is defined to be*

$$G_R(A) :=\{g \in A | (g \otimes g)\Delta(\mathbf{1}) = \Delta(g) = \Delta(\mathbf{1})(g \otimes \Pi^L(g)^{-1}g); \Pi^{R/L}(g) \in A_*^{R/L}\} \tag{5.1a}$$

$$G_L(A) :=\{g \in A | (g\Pi^R(g)^{-1} \otimes g)\Delta(\mathbf{1}) = \Delta(g) = \Delta(\mathbf{1})(g \otimes g); \Pi^{R/L}(g) \in A_*^{R/L}\} \tag{5.1b}$$

where $A_*^{R/L}$ denote the set of invertible elements in $A^{R/L}$. The set of grouplike elements in A is defined to be the intersection $G(A) := G_R(A) \cap G_L(A)$.

Using the form (1.3), (1.10) of the maps $\Pi^{R/L}$ and the identities (1.8) the defining properties for $g \in G_{R/L}(A)$ lead to

$$\mathbf{1} =\Pi^R(g) = S(g)\Pi^L(g)^{-1}g, \qquad \Pi^L(g) = gS(g); \qquad g \in G_R(A), \tag{5.2a}$$

$$\mathbf{1} =\Pi^L(g) = g\Pi^R(g)^{-1}S(g), \qquad \Pi^R(g) = S(g)g; \qquad g \in G_L(A), \tag{5.2b}$$

hence elements of $G_{R/L}(A)$ are themselves invertible. Using (5.1–2) it is easy to show that $G_R(A)$ and $G_L(A)$, hence $G(A)$ too, are groups, $G_L(A) = S(G_R(A))$, and the definition of grouplike elements $G(A)$ above is equivalent to that of given in [2]. We note that the set $G(A)$ in $G_R(A)$ can also be given by the subset of elements satisfying $\Pi^L(g) = \mathbf{1}$ or by the subset of pointwise invariant elements with respect to S^2. For verification of the latter claim we note that if $g = S^2(g)$ holds for $g \in G_R(A)$ then $\Pi^L(g) = gS(g) = S^2(g)S(g) = S(\Pi^L(g))$, that is $\Pi^L(g)$, hence $\Pi^L(g)^{-1}$, too, are in $A^L \cap A^R \subset \mathrm{Center}\, A^L$. Using these consequences for $g^{-1} = S^2(g^{-1})$ and (5.1a) one obtains

$$\begin{aligned}\mathbf{1} = \Pi^R(g) &= S(\mathbf{1}^{(1)})S(g)g\mathbf{1}^{(2)} = S(\mathbf{1}^{(1)})\Pi^L(g^{-1})^{-1}\mathbf{1}^{(2)} \\ &= S(\mathbf{1}^{(1)})\mathbf{1}^{(2)}\Pi^L(g^{-1})^{-1} = S(g)g.\end{aligned}$$

In order to reveal the meaning of (left/right) grouplike elements in the representation category of the dual weak Hopf algebra $\hat{A}$ we give the following

Definition 5.2 *An object M of a monoidal category $(\mathcal{L}; \times, E)$ is invertible if there exists an inverse object $\bar{M} \in \mathrm{Obj}\,\mathcal{L}$ obeying $M \times \bar{M} \simeq E \simeq \bar{M} \times M$, where $E \in \mathrm{Obj}\,\mathcal{L}$ is the monoidal unit of the category and $\simeq$ denotes equivalence of objects in $\mathcal{L}$.*[1]

Lemma 5.3 *i) Let $\mathcal{L}$ be the autonomous monoidal category of finite dimensional left (right) A-modules of a WHA A given in Prop. 2.2. The left (right) A-module $M \in \mathrm{Obj}\,\mathcal{L}$ is invertible iff it becomes a free left (right) A^L- and A^R-module of rank 1 by restricting A to the subalgebras A^L and A^R, respectively.*

ii) An invertible left/right A-module $M \in \mathrm{Obj}\,\mathcal{L}$ is the direct sum of inequivalent indecomposable submodules:

$$\begin{aligned}{}_AM &= \oplus M_{(\tau_M(p),p)}, \qquad & M_{(\tau_M(p),p)} &:= z^L_{\tau_M(p)} \cdot M = z^R_p \cdot M, \\ M_A &= \oplus M_{(\tau_M(p),p)}, \qquad & M_{(\tau_M(p),p)} &:= M \cdot z^L_{\tau_M(p)} = M \cdot z^R_p,\end{aligned}$$

where $\{z^L_p\}_p \subset Z^L$ and $\{z^R_p := S(z^L_p)\}_p \subset Z^R$ are complete set of primitive idempotents and τ_M is a permutation on them.

Proof: i) First we show that the left A-module M is invertible iff

$$M \times \overleftarrow{M} \simeq A^L \simeq \overrightarrow{M} \times M \tag{5.3}$$

as left A-modules.

Clearly, if (5.3) holds then using the natural equivalences X^L and X^R given in (2.3) it follows that $\overleftarrow{M} \simeq A^L \times \overleftarrow{M} \simeq \overrightarrow{M} \times M \times \overleftarrow{M} \simeq \overrightarrow{M} \times A^L \simeq \overrightarrow{M}$, hence M is invertible. Now, let M be invertible with inverse $\bar{M}$ and let $\sigma\colon \bar{M} \times M \to A^L$ and $\tau\colon M \times \bar{M} \to A^L$ be the corresponding invertible arrows. We will show that $\overleftarrow{M} \simeq \bar{M} \simeq \overrightarrow{M}$, which imply (5.3). The arrow

$$\omega := X^{L\,-1}_{A^L}(\tau \times \tau)(1_M \times \sigma^{-1} \times 1_{\bar{M}})(X^R_M \times 1_{\bar{M}})\tau^{-1} \in \mathrm{End}\,{}_AA^L \tag{5.4}$$

[1] In case of symmetric or braided monoidal categories invertibility is defined by the condition $E \simeq M \times \bar{M}$ [7].

is invertible, therefore it is given by the action of an invertible element $z^L \in Z^L := A^L \cap$ Center A due to End ${}_A A^L = Z^L\cdot$ proved in Theorem 2.4. Hence, if $z^L_N\colon N \to N$ denotes the arrow given by the action $z^L \in Z^L$ for $N \in \mathrm{Obj}\,\mathcal{L}$ then $\{z^L_N\}_N$ is a natural automorphism of the identity functor on $\mathcal{L}$ and $\omega = z^L_{A^L}$. Defining $\tilde\tau := z^{L\,-1}_{A^L}\tau = \tau(z^{L\,-1}_M \times 1_{\bar M})\colon M \times \bar M \to A^L$ and using the form (5.4) of ω the identity $z^{L\,-1}_{A^L}\omega = 1_{A^L}$ leads to

$$X^{L\,-1}_M(\tilde\tau \times 1_M)(1_M \times \sigma^{-1})X^R_M = 1_M. \tag{5.5a}$$

Hence,

$$X^{R\,-1}_{\bar M}(1_{\bar M} \times \tilde\tau)(\sigma^{-1} \times 1_{\bar M})X^L_{\bar M} = 1_{\bar M} \tag{5.5b}$$

also holds because of the identity

$$\begin{aligned} 1_{A^L} &= \tilde\tau\tilde\tau^{-1} = \tilde\tau(X^{L\,-1}_M \times 1_{\bar M})(\tilde\tau \times 1_M \times 1_{\bar M})(1_M \times \sigma^{-1} \times 1_{\bar M})(X^R_M \times 1_{\bar M})\tilde\tau^{-1} \\ &= X^{L\,-1}_{A^L}(\tilde\tau \times \tilde\tau)(1_M \times \sigma^{-1} \times 1_{\bar M})(X^R_M \times 1_{\bar M})\tilde\tau^{-1} \\ &= \tilde\tau[1_M \times X^{R\,-1}_{\bar M}(1_{\bar M} \times \tilde\tau)(\sigma^{-1} \times 1_{\bar M})X^L_{\bar M}]\tilde\tau^{-1}. \end{aligned} \tag{5.6}$$

Therefore using the right and left evaluation maps defined in (2.8) and (2.14)

$$X^{L\,-1}_{\bar M}(E^l_M \times 1_{\bar M})(1_{\overleftarrow{M}} \times \tilde\tau^{-1})X^R_{\overleftarrow{M}}\colon \overleftarrow{M} \to \bar M, \tag{5.7a}$$

$$X^{R\,-1}_{\bar M}(1_{\bar M} \times E^r_M)(\sigma^{-1} \times 1_{\overrightarrow{M}})X^L_{\overrightarrow{M}}\colon \overrightarrow{M} \to \bar M \tag{5.7b}$$

provide the equivalences $\overleftarrow{M} \simeq \bar M \simeq \overrightarrow{M}$ with the inverse arrows

$$X^{L\,-1}_{\overleftarrow{M}}(\sigma \times 1_{\overleftarrow{M}})(1_{\bar M} \times C^l_M)X^R_{\bar M}\colon \bar M \to \overleftarrow{M}, \tag{5.8a}$$

$$X^{R\,-1}_{\overrightarrow{M}}(1_{\overrightarrow{M}} \times \tilde\tau)(C^r_M \times 1_{\bar M})X^L_{\bar M}\colon \bar M \to \overrightarrow{M} \tag{5.8b}$$

due to the left and right rigidity identities (2.10) and (2.15), respectively, and due to (5.5a–b).

Now we prove that (5.3) fulfills iff M becomes a free left A^L- and A^R-module with a single generator by restricting the left A-action to these subalgebras. Let $V\colon A^L \to M \times \overleftarrow{M}$ and $W\colon A^L \to \overrightarrow{M} \times M$ be the invertible arrows required by (5.3). Realizing $\mathrm{End}_k M$ by $M \otimes \hat M$ one proves similarly to (4.12–13) that End ${}_{A^R}M = M \times \overleftarrow{M}$ and End ${}_{A^L}M = \overrightarrow{M} \times M$ as k-linear spaces. Let $\{m_i\}_i \subset M$ and $\{\hat m_i\}_i \subset \hat M$ denote dual bases then

$$\mathrm{End}\,{}_{A^R}M = M \times \overleftarrow{M} = V(A^L) = V(A^L \cdot \mathbf{1}) = A^L \cdot V(\mathbf{1}) = A^L \cdot m_i \otimes \hat m_i, \tag{5.9a}$$

$$\mathrm{End}\,{}_{A^L}M = \overrightarrow{M} \times M = W(A^L) = W(A^R \cdot \mathbf{1}) = A^R \cdot W(\mathbf{1}) = \hat m_i \otimes A^R \cdot m_i, \tag{5.9b}$$

imply that

$$A^L \simeq \mathrm{End}\,{}_{A^R}M = A^L\,\cdot \qquad A^R \simeq \mathrm{End}\,{}_{A^L}M = A^R\cdot \tag{5.10}$$

as k-algebras. Therefore, as a left A^T-module M is a direct sum of simple A^T-submodules

$$M = \bigoplus_i M_{\sigma_M(i)i}, \qquad M_{\sigma_M(i)i} := e^L_{\sigma_M(i)} e^R_i \cdot M, \tag{5.11}$$

where $\{e_i^{R/L}\}_i \subset A^{R/L}$ are complete set of primitive central idempotents and σ_M is a permutation of them by an argument similar to that of around (4.18). Then by repeating the argument given around (4.10–11) one proves that M is a free left A^L- and A^R-module with a single generator.

Conversely, let ${}_AM$ become a free left A^L- and A^R-module with a single generator $m \in M$ by resricting the A-action to these subalgebras. The elements $\hat{m}_l$ and $\hat{m}_r$ of the k-dual $\hat{M}$ of M defined by

$$\langle \hat{m}_l, x^R \cdot m\rangle_M := \varepsilon(x^R), \qquad \langle \hat{m}_r, x^L \cdot m\rangle_M := \varepsilon(x^L), \qquad x^{R/L} \in A^{R/L} \tag{5.12}$$

are $A^{L/R}$-generators of $\overleftarrow{M}$ and $\overrightarrow{M}$, respectively, because choosing dual bases $\{e_i\}_i, \{f_i\}_i \subset A^L$ with respect to the counit ε the bases $\{e_i \cdot \hat{m}_l\}_i \subset \overleftarrow{M}, \{S^{-1}(f_i) \cdot m\}_i \subset M$ and $\{S^{-1}(f_i) \cdot \hat{m}_r\}_i \subset \overrightarrow{M}, \{e_i \cdot m\}_i \subset M$ become dual to each other. Indeed, for the dual A^L-bases we have

$$\delta_{ij} = \varepsilon(e_i f_j) = \varepsilon(f_j S^2(e_i)) = \varepsilon(S(e_i)S^{-1}(f_j)). \tag{5.13}$$

The third equality follows from the invariance of the counit with respect to the antipode: $\varepsilon = \varepsilon \circ S$, while the second is the consequence of the identities (1.14–15) claiming that S^2 is the Nakayama automorphism $\theta_L\colon A^L \to A^L$ corresponding to the counit as a non-degenerate functional on A^L. Therefore

$$\begin{aligned} \delta_{ij} &= \varepsilon(S(e_i)S^{-1}(f_j)) := \langle \hat{m}_l, S(e_i)S^{-1}(f_j) \cdot m\rangle_M \\ &= \langle e_i \cdot \hat{m}_l, S^{-1}(f_j) \cdot m\rangle_M \,, \\ \delta_{ij} &= \varepsilon(S^{-2}(f_j)e_i) := \langle \hat{m}_r, S^{-2}(f_j)e_i \cdot m\rangle_M \\ &= \langle S^{-1}(f_j) \cdot \hat{m}_r, e_i \cdot m\rangle_M \,. \end{aligned} \tag{5.14}$$

Now we prove that the left and right coevaluation maps $C^l_M\colon A^L \to M \times \overleftarrow{M}$ and $C^r_M\colon A^L \to \overrightarrow{M} \times M$ defined in (2.8) and (2.14) are invertible, that is (5.3) holds. Using that $\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} = S^{-1}(f_i) \otimes e_i$ (summation suppressed) in terms of the dual bases in A^L one has

$$\begin{aligned} M \times \overleftarrow{M} &:= \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} \cdot \overleftarrow{M} = \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} A^L \cdot \hat{m}_l \\ &= \mathbf{1}^{(1)} S^{-1}(A^L) \cdot M \otimes \mathbf{1}^{(2)} \cdot \hat{m}_l = \mathbf{1}^{(1)} A^L \cdot m \otimes \mathbf{1}^{(2)} \cdot \hat{m}_l \\ &= A^L S^{-1}(f_i) \cdot m \otimes e_i \cdot \hat{m}_l = C^l_M(A^L), \\ \overrightarrow{M} \times M &:= \mathbf{1}^{(1)} \cdot \overrightarrow{M} \otimes \mathbf{1}^{(2)} \cdot M = \mathbf{1}^{(1)} \cdot A^R \hat{m}_r \otimes \mathbf{1}^{(2)} \cdot M \\ &= \mathbf{1}^{(1)} \cdot \hat{m}_r \otimes \mathbf{1}^{(2)} S^{-1}(A^L) \cdot M = \mathbf{1}^{(1)} \cdot \hat{m}_r \otimes \mathbf{1}^{(2)} A^R \cdot m \\ &= S^{-1}(f_i) \cdot \hat{m}_r \otimes A^R e_i \cdot m = A^L S^{-1}(f_i) \cdot \hat{m}_r \otimes e_i \cdot m = C^l_M(A^L), \end{aligned} \tag{5.15}$$

i.e. C^l_M and C^r_M are surjective. Injectivity of C^l_M and C^r_M follow from the faithfulness of M as a left A^L- and A^R-module, respectively. The inverse maps $x^L S^{-1}(f_i)\cdot m\otimes e_i\cdot\hat{m}_l\to x^L$ and $x^L S^{-1}(f_i)\cdot\hat{m}_r\otimes e_i\cdot m\to x^L$, where $x^L\in A^L$ are left A-module maps due to the properties (2.7a) and (2.13a) of the dual bases and their explicit form are as follows:

$$\begin{aligned} C^{l-1}_M(n\otimes\hat{n}) &= S(\mathbf{1}^{(1)})\langle\hat{m}_r,\mathbf{1}^{(2)}\cdot n\rangle_M\langle\hat{n},m\rangle_M, \qquad n\otimes\hat{n}\in M\times\overleftarrow{M},\\ C^{r-1}_M(\hat{n}\otimes n) &= \mathbf{1}^{(2)}\langle m,\mathbf{1}^{(2)}\cdot\hat{n}\rangle_M\langle n,\hat{m}_l\rangle_M, \qquad \hat{n}\otimes n\in\overrightarrow{M}\times M. \end{aligned} \tag{5.16}$$

The case of invertible right A-modules is analogous.

ii) From the $A^{L/R}$-freeness of M and from (5.10) we can deduce that

$$\mathrm{End}_A M\subset\mathrm{End}_{A^R}M\cap\mathrm{End}_{A^L}M=(\mathrm{Center}\,A^L)\cdot=(\mathrm{Center}\,A^R)\cdot\ . \tag{5.17}$$

Let $m\in M$ be a free A^L-generator. The action by an element $x^L\in\mathrm{Center}\,A^L$ on M represents an element of $\mathrm{End}_A M$ only if

$$\Pi^L(a)x^L\cdot m=a^{(1)}S(a^{(2)})x^L\cdot m=a^{(1)}x^L S(a^{(2)})\cdot m=\Pi^L(ax^L)\cdot m,\qquad a\in A, \tag{5.18}$$

i.e. only if $\Pi^L(a)x^L=\Pi^L(ax^L)$ for all $a\in A$. Applying (2.24) with $f(\mathbf{1})=x^L$ one obtains that $x^L\in Z^L$, that is $\mathrm{End}_A M\subset Z^L$. The opposite containment is trivial. Hence, the indecomposability of the direct summands $z^L_p\cdot M$ of M follows, because $\mathrm{End}_A(z^L_p\cdot M)=z^L_p Z^L\cdot$ is an (Abelian) division algebra. Similarly one proves that $\mathrm{End}_A(z^R_p\cdot M)=z^R_p Z^R\cdot$ and the permutation τ_M is induced by the permutation σ_M of the subprojections $\{e^L_i\}_i\subset\mathrm{Center}\,A^L$ of the idempotents $z^L_p=\sum_{i\in p}e^L_i\in Z^L:=\mathrm{Center}\,A\cap A^L$ through (5.11).

Inequivalence of the different direct summands $z^L_p\cdot M$ follows from the orthogonality, $z^L_p z^L_q=\delta_{pq}z^L_p$, of the primitive idempotents $\{z^L_p\}_p\subset Z^L$.

The case of invertible right A-modules is analogous. ■

The connection between various types of grouplike elements in a WHA A and invertible left (or right) $\hat{A}$-modules of the dual WHA $\hat{A}$ can be formulated as follows:

Proposition 5.4 *Let A be a WHA and let $F_a:=(\hat{A}\rightharpoonup a,\rightharpoonup)$ denote the principal left $\hat{A}$-submodule of ${}_{\hat{A}}A:={}_{\hat{A}}(A,\rightharpoonup)$ generated by $a\in A$.*

i) The element $g\in A$ is (right/left) grouplike iff g is an element of an invertible submodule ${}_{\hat{A}}F$ of ${}_{\hat{A}}A$ and g obeys the normalization conditions ($\Pi^{R/L}(g)=\mathbf{1}$) $\Pi^R(g)=\mathbf{1}=\Pi^L(g)$.

ii) Two (right/left) grouplike elements g,h lead to equivalent submodules $F_g,F_h\subset{}_{\hat{A}}A$ iff gh^{-1} is an element of the trivial subalgebra A^T.
The elements of $G^T_{R/L}(A):=G_{R/L}(A)\cap A^T$ are of the form $g_L S(g_L^{-1})\in G^T_R(A)$ and $g_L S^{-1}(g_L^{-1})\in G^T_L(A)$ with $g_L\in A^L$ invertible and they form a normal subgroup in $G_{R/L}(A)$.
The elements of $G^T(A):=G(A)\cap A^T$ are of the form $g_L S(g_L^{-1})$ with $g_L=S^2(g_L)\in A^L$ invertible and they form a normal subgroup in $G(A)$.

iii) For any invertible submodule of ${}_{\hat{A}}A$ there is an equivalent submodule of ${}_{\hat{A}}A$ which contains a right (left) grouplike element $g\in G_R(A)$ ($g\in G_L(A)$).

iv) Every invertible left $\hat{A}$-module is semisimple and isomorphic to a principal submodule $F_g \subset {}_{\hat{A}}A$ with $g \in G_R(A)$ ($g \in G_L(A)$). The inequivalent invertible left $\hat{A}$-modules can be characterized by the elements of the (finite) factor group $G_R(A)/G^T_R(A)$ ($G_L(A)/G^T_L(A)$).

Proof: i) Let $g \in G_{R/L}(A)$ or $g \in G(A)$. Clearly, F_g is a submodule of ${}_{\hat{A}}A$ that contains g satisfying the required normalization conditions. According to Lemma 5.3 i) invertibility of F_g follows if F_g becomes a free left $\hat{A}^L$- and $\hat{A}^R$-module with the single generator g by restricting the left $\hat{A}$-action to the corresponding subalgebras. If $g \in G_R(A)$ then the identities (1.6–7), (1.11) and (5.1–2a) lead to the relations

$$\begin{aligned}\varphi \rightharpoonup g &= \mathbf{1}^{(1)} g \langle \varphi, \mathbf{1}^{(2)} S(g)^{-1} \rangle = ((S(g)^{-1} \rightharpoonup \varphi) \rightharpoonup \mathbf{1}) g \\ &= (\hat{\Pi}^L(S(g)^{-1} \rightharpoonup \varphi) \rightharpoonup \mathbf{1}) g = \hat{\Pi}^L(S(g)^{-1} \rightharpoonup \varphi) \rightharpoonup g, \qquad \varphi \in \hat{A}, \quad (5.19a) \\ \varphi \rightharpoonup g &= g \mathbf{1}^{(1)} \langle \varphi, g \mathbf{1}^{(2)} \rangle = g((\varphi \leftharpoonup g) \rightharpoonup \mathbf{1}) \\ &= g(\hat{\bar{\Pi}}^R(\varphi \leftharpoonup g) \rightharpoonup \mathbf{1}) = \hat{\bar{\Pi}}^R(\varphi \leftharpoonup g) \rightharpoonup g, \qquad \varphi \in \hat{A}. \quad (5.19b)\end{aligned}$$

They imply that $F_g \subset (\hat{A}^R \rightharpoonup g) \cap (\hat{A}^L \rightharpoonup g) = gA^R \cap A^R g$, moreover, if $0 = \varphi^L \rightharpoonup g = (\varphi^L \rightharpoonup \mathbf{1})g$ or $0 = \varphi^R \rightharpoonup g = g(S(\varphi^L) \rightharpoonup \mathbf{1})$ for certain $\varphi^{R/L} \in \hat{A}^{R/L}$ then $\varphi^{R/L} = 0$, because g is invertible and the maps $\hat{\kappa}_L$ in (1.5) and the antipode S are bijections. Therefore F_g is a free left $\hat{A}^R$- and $\hat{A}^L$-module with a single generator g for any $g \in G_R(A)$, i.e. F_g is invertible for any $g \in G_R(A)$. The case of $g \in G_L(A)$, hence the case $g \in G(A)$, too, can be proved similarly if one writes the first coproduct form for $g \in G_L(A)$ as

$$\Delta(g) = (g\Pi^R(g)^{-1} \otimes g)\Delta(\mathbf{1}) = (g \otimes gS^{-1}(\Pi^R(g)^{-1}))\Delta(\mathbf{1}) = (g \otimes S^{-1}(g^{-1}))\Delta(\mathbf{1}). \quad (5.20)$$

Conversely, let ${}_{\hat{A}}F$ be an invertible submodule of ${}_{\hat{A}}A$. Then F is a right coideal in A and a free left $\hat{A}^L$- and $\hat{A}^R$-module with a single generator $f \in F$ by restricting the left $\hat{A}$-action to these subalgebras. Thus one can define two projections $\Phi^L_f \colon \hat{A} \to \hat{A}^L$ and $\bar{\Phi}^R_f \colon \hat{A} \to \hat{A}^R$ by requiring

$$\Phi^L_f(\varphi) \rightharpoonup f := \varphi \rightharpoonup f, \qquad \bar{\Phi}^R_f(\varphi) \rightharpoonup f := \varphi \rightharpoonup f, \quad (5.21)$$

for $\varphi \in \hat{A}$. They are left $\hat{A}^L$- and $\hat{A}^R$-module maps, respectively, and defining $\hat{f}_l$ and $\hat{f}_r$ in the k-dual $\hat{F}$ of F as in (5.12) by

$$\begin{aligned}\langle \hat{f}_r, \varphi^L \rightharpoonup f \rangle_F &= \langle f \leftharpoonup \hat{f}_r, \varphi^L \rangle := \hat{\varepsilon}(\varphi^L), \qquad \varphi^L \in \hat{A}^L, \\ \langle \hat{f}_l, \varphi^R \rightharpoonup f \rangle_F &= \langle f \leftharpoonup \hat{f}_l, \varphi^R \rangle := \hat{\varepsilon}(\varphi^R), \qquad \varphi^R \in \hat{A}^R,\end{aligned} \quad (5.22)$$

we have $\Pi^L(f \leftharpoonup \hat{f}_r) = \mathbf{1} = \Pi^R(f \leftharpoonup \hat{f}_l)$ and

$$\begin{aligned}\Phi^L_f(\varphi) &= \hat{S}(\hat{\mathbf{1}}^{(1)}) \langle \hat{f}_r, \hat{\mathbf{1}}^{(2)} \varphi \rightharpoonup f \rangle_F = \hat{S}(\hat{\mathbf{1}}^{(1)}) \langle f \leftharpoonup \hat{f}_r, \hat{\mathbf{1}}^{(2)} \varphi \rangle \\ &= \hat{\Pi}^L(\varphi^{(1)}) \langle f \leftharpoonup \hat{f}_r, \varphi^{(2)} \rangle, \\ \bar{\Phi}^R_f(\varphi) &= \langle \hat{f}_l, \hat{\mathbf{1}}^{(1)} \varphi \rightharpoonup f \rangle_F \hat{S}^{-1}(\hat{\mathbf{1}}^{(2)}) = \langle f \leftharpoonup \hat{f}_l, \hat{\mathbf{1}}^{(1)} \varphi \rangle \hat{S}^{-1}(\hat{\mathbf{1}}^{(2)}) \\ &= \langle f \leftharpoonup \hat{f}_l, \varphi^{(1)} \rangle \hat{\bar{\Pi}}^R(\varphi^{(2)}).\end{aligned} \quad (5.23)$$

Therefore using the identities (1.7–8)

$$\begin{aligned}\varphi \rightharpoonup f = \Phi^L_f(\varphi) \rightharpoonup f &= f^{(1)}\langle \hat{\Pi}^L(\varphi^{(1)}), f^{(2)}\rangle\langle f \leftharpoonup \hat{f}_r, \varphi^{(2)}\rangle \\ &= f^{(1)}\langle \varphi, \Pi^L(f^{(2)})(f \leftharpoonup \hat{f}_r)\rangle = \mathbf{1}^{(1)} f\langle \varphi, \mathbf{1}^{(2)}(f \leftharpoonup \hat{f}_r)\rangle, \qquad (5.24a)\\ \varphi \rightharpoonup f = \bar{\Phi}^R_f(\varphi) \rightharpoonup f &= f^{(1)}\langle \varphi^{(1)}, f \leftharpoonup \hat{f}_l\rangle\langle \hat{\bar{\Pi}}^R(\varphi^{(2)}), f^{(2)}\rangle \\ &= f^{(1)}\langle \varphi, (f \leftharpoonup \hat{f}_l)\bar{\Pi}^L(f^{(2)})\rangle = f\mathbf{1}^{(1)}\langle \varphi, (f \leftharpoonup \hat{f}_l)\mathbf{1}^{(2)}\rangle, \qquad (5.24b)\end{aligned}$$

for all $\varphi \in \hat{A}$, which imply

$$\mathbf{1}^{(1)} f \otimes \mathbf{1}^{(2)}(f \leftharpoonup \hat{f}_r) = f^{(1)} \otimes f^{(2)} = f\mathbf{1}^{(1)} \otimes (f \leftharpoonup \hat{f}_l)\mathbf{1}^{(2)}. \qquad (5.25)$$

Applying the counit ε to the first tensor factor we obtain

$$\Pi^L(f)(f \leftharpoonup \hat{f}_r) = f = (f \leftharpoonup \hat{f}_l)\bar{\Pi}^L(f). \qquad (5.26)$$

Let g be the element of F that obeys $\Pi^R(g) = \mathbf{1}$. Then $g = \varphi^R \rightharpoonup f = f(\varphi^R \rightharpoonup \mathbf{1}) =: fx^R$ for some $\varphi^R \in \hat{A}^R$ due to the $\hat{A}^R$-freeness of F and (1.6). Therefore $\mathbf{1} = \Pi^R(g) = \Pi^R(f)x^R$, that is x^R is invertible, which implies that g is also an $\hat{A}^{L/R}$-generator of F. Thus (5.25–26) hold for $f = g \in F$, too. Since $\mathbf{1} = S^{-1}(\Pi^R(g)) = \bar{\Pi}^L(g)$ by assumption equation (5.26) implies that $g \leftharpoonup \hat{g}_l = g$, hence $\Pi^L(g) = gS(g)$ due to (5.25). However, $(\mathbf{1} =)\Pi^R(g) = S(g)(g \leftharpoonup \hat{g}_r)$ also holds due to (5.25), therefore $S(g)$, hence g and $\Pi^L(g) = gS(g)$ are invertible, i.e. $g \leftharpoonup \hat{g}_r = \Pi^L(g)^{-1}g$ due to (5.26). The substitution of these results into (5.25) leads to the coproduct property (5.1a) of a right grouplike element g in A. The cases $g \in G_L(A), G(A)$ can be proved similarly.

ii) First we note that for $g, h \in G_R(A)$ $(G_L(A), G(A))$ the invertible left $\hat{A}$-modules F_{gh} and $F_g \times F_h$ are equivalent, because the maps

$$\begin{aligned} U\colon F_g \times F_h &\to F_{gh} & V\colon F_{gh} &\to F_g \times F_h \\ m \otimes n &\mapsto mn & m &\mapsto \hat{\mathbf{1}}^{(1)} \rightharpoonup mh^{-1} \otimes \hat{\mathbf{1}}^{(2)} \rightharpoonup h \end{aligned} \qquad (5.27)$$

are left $\hat{A}$-module maps, which are inverses of each other. Since the unit element $\mathbf{1}$ of A is grouplike and since the invertible left $\hat{A}$-module $F_{\mathbf{1}}$ is equivalent to the left unit $\hat{A}$-module ${}_{\hat{A}}\hat{A}^L$ through the maps $\kappa_R\colon A^R \to \hat{A}^L$ and $\hat{\kappa}_L\colon \hat{A}^L \to A^R$ defined in (1.5) it is enough to prove that $F_g \simeq F_{\mathbf{1}}$ as left $\hat{A}$-modules for a $g \in G_R(A)$ $(G_L(A), G(A))$ iff $g \in A^T$.

Let $g \in G^T_R(A)$ $(G^T_L(A), G^T(A))$. Then the ideal $(A^T)^\perp := \{\varphi \in \hat{A} | \langle \varphi, A^T\rangle = 0\} \subset \hat{A}$ is in the annihilator ideal of both of the left $\hat{A}$-modules $F_{\mathbf{1}}$ and F_g, because $F_{\mathbf{1}}, F_g \subset A^T$ and A^T is a subcoalgebra of A. Therefore $F_{\mathbf{1}}$ and F_g are also left modules with respect to the factor algebra $\hat{A}/(A^T)^\perp$ and the equivalence of the modules $F_{\mathbf{1}}$ and F_g with respect to this factor algebra ensures their equivalence as $\hat{A}$-modules. The factor algebra $\hat{A}/(A^T)^\perp$ is isomorphic to the dual WHA $\widehat{A^T}$ of A^T as an algebra, which is isomorphic to a direct sum of simple matrix algebras, $\widehat{A^T} \simeq \oplus_\alpha M_{n_\alpha}(Z_\alpha)$, due to Lemma 2.3, where the Z_αs are separable field extensions of the ground field k determined by the ideal decomposition $Z = \oplus_\alpha Z_\alpha$ of

$Z \equiv A^L \cap A^R$ and the dimensions obey $n_\alpha = \dim_{Z_\alpha} A^L_\alpha$. Hence, $F_{\mathbf{1}}$ and F_g are equivalent $\widehat{A^T}$-modules if the multiplicities corresponding to the Wedderburn components of $\widehat{A^T}$ in their direct sum decompositions are equal. In order to prove this, first we note that the primitive idempotents $\{z_\alpha\}_\alpha \subset Z$ are central in A^T, hence they are in the hypercenter H of A^T and they are related to the primitive central idempotents $\{\hat{e}_\alpha\}_\alpha$ of $\widehat{A^T}$ as

$$\hat{e}_\alpha \rightharpoonup \mathbf{1} = z_\alpha = \mathbf{1} \leftharpoonup \hat{e}_\alpha \tag{5.28}$$

due to (1.6) and remarks after eq. (1.8). Hence, $\hat{e}_\alpha \rightharpoonup g = (\hat{e}_\alpha \rightharpoonup \mathbf{1})g = z_\alpha g$ and $F_{\mathbf{1}}$ and F_g are faithful left $\widehat{A^T}$-modules, because $\mathbf{1}$ and g are invertible. Therefore the multiplicity corresponding to a Wedderburn component of $\widehat{A^T}$ is at least one in both of the modules $F_{\mathbf{1}}$ and F_g. Then the identity

$$|F_{\mathbf{1}}| = |F_g| = |\hat{A}^R| = |A^L| = \sum_\alpha |Z_\alpha| \dim_{Z_\alpha} A^L_\alpha = \sum_\alpha |Z_\alpha| n_\alpha \tag{5.29}$$

for k-dimensions coming from the $\hat{A}^R$-freeness of invertible $\hat{A}$-modules and from the algebra structure of $\widehat{A^T}$ ensures that these multiplicities are equal to one, that is $F_{\mathbf{1}}$ and F_g are equivalent $\widehat{A^T} \simeq \hat{A}/(A^T)^\perp$, hence equivalent $\hat{A}$-modules.

Conversely, let $g \in G_R(A)$ $(G_L(A), G(A))$ be such that there exists a $U\colon F_{\mathbf{1}} \to F_g$ equivalence between the invertible left $\hat{A}$-modules $F_{\mathbf{1}}$ and F_g. Using that U is an $\hat{A}$-module map we have

$$\begin{aligned} U(\mathbf{1})^{(1)}\langle\varphi, U(\mathbf{1})^{(2)}\rangle &= \varphi \rightharpoonup U(\mathbf{1}) = U(\varphi \rightharpoonup \mathbf{1}) = U(\hat{\Pi}^L(\varphi) \rightharpoonup \mathbf{1}) \\ &= \hat{\Pi}^L(\varphi) \rightharpoonup U(\mathbf{1}) = U(\mathbf{1})^{(1)}\langle\hat{\Pi}^L(\varphi), U(\mathbf{1})^{(2)}\rangle \\ &= U(\mathbf{1})^{(1)}\langle\varphi, \Pi^L(U(\mathbf{1})^{(2)})\rangle = \mathbf{1}^{(1)}U(\mathbf{1})\langle\varphi, \mathbf{1}^{(2)}\rangle, \qquad \varphi \in \hat{A}, \end{aligned} \tag{5.30}$$

that is $\Delta(\mathbf{1}) = \mathbf{1}^{(1)}U(\mathbf{1}) \otimes \mathbf{1}^{(2)}$, which ensures that $U(\mathbf{1}) \in A^L$. Moreover, $U(\mathbf{1})$ is an $\hat{A}^{L/R}$ generator of F_g, because it is the image of the $\hat{A}^{L/R}$ generator $\mathbf{1} \in F_{\mathbf{1}}$. Hence, there exists an invertible element $\varphi^L \in \hat{A}^L$ such that $g = \varphi^L \rightharpoonup U(\mathbf{1}) = (\varphi^L \rightharpoonup \mathbf{1})U(\mathbf{1}) \in A^R A^L$, that is $g \in A^T$.

The very last claim already implies that an element $g \in G^T_R(A)$ $(G^T_L(A), G^T(A))$ has the product form $g = g_L g_R$ with $g_L := U(\mathbf{1}) \in A^L$ and $g_R := \varphi^L \rightharpoonup \mathbf{1} \in A^R$. Since g is invertible g_L and g_R are invertible. Using property (5.2) one obtains $\mathbf{1} = \Pi^R(g) \equiv \Pi^R(g_L g_R) = g_R S(g_L)$ for $g \in G^T_R(A)$ and $\mathbf{1} = \Pi^L(g) \equiv \Pi^L(g_L g_R) = g_L S(g_R)$ for $g \in G^T_L(A)$. Since $G^T(A) = G^T_R(A) \cap G^T_L(A)$ the form $g = g_L S(g_L^{-1})$ with $S^2(g_L) = g_L$ for $g \in G^T(A)$ is also proven.

Since $\hat{A} \rightharpoonup g \leftharpoonup \hat{A} = gA^T = A^T g$ for $g \in G_R(A)$ $(G_L(A), G(A))$ due to (5.1) it follows that $gA^T g^{-1} = A^T$. Therefore $G^T_R(A)$ $(G^T_L(A), G^T(A))$ is a normal subgroup in $G_R(A)$ $(G_L(A), G(A))$.

iii) Let f be an $\hat{A}^{L/R}$-generator of the invertible submodule $F_f \subset {}_{\hat{A}}A$. If there is no such element g in F_f that obeys $\Pi^R(g) = \mathbf{1}$ let us define $g := f \leftharpoonup \hat{f}_l \in A$ with $\hat{f}_l$ given in

(5.22). The maps

$$\leftharpoonup \hat{f}_l : F_f \to F_g, \tag{5.31a}$$
$$x_R f \mapsto x_R f \leftharpoonup \hat{f}_l = x_R (f \leftharpoonup \hat{f}_l) = x_R g$$
$$\leftharpoonup (\bar{\Pi}^L(f) \rightharpoonup \hat{\mathbf{1}}) : F_g \to F_f \tag{5.31b}$$
$$x_R g \mapsto x_R g \leftharpoonup (\bar{\Pi}^L(f) \rightharpoonup \hat{\mathbf{1}}) = x_R g \bar{\Pi}^L(f) = x_R f$$

commute with the left Sweedler action, i.e. they are left $\hat{A}$-module maps, and they are inverses of each other due to (5.26), which property has been already indicated in (5.31a–b). Therefore F_g and F_f are equivalent submodules of ${}_{\hat{A}}A$, that is F_g is also invertible. Since $\Pi^R(g) \equiv \Pi^R(f \leftharpoonup \hat{f}_l) = \mathbf{1}$ due to (5.22) and due to the non-degeneracy of the $A^R - \hat{A}^R$ pairing g is a right grouplike element due to i). The proof is similar for left grouplike elements, one has to define $g := f \leftharpoonup \hat{f}_r$ with $\hat{f}_r$ given in (5.22) to get $g \in G_L(A)$ in the submodule F_g equivalent to F_f.

iv) Lemma 5.3 ii) claims that an invertible left $\hat{A}$-module M is the direct sum of inequivalent indecomposable submodules: $M = \oplus_p \hat{z}^R_p \cdot M \equiv \oplus_p M_p$, where $\{\hat{z}^R_p\}_p \subset \hat{Z}^R$ is a complete set of primitive idempotents. Since $\hat{A}$ is a Frobenius algebra (see Corollary 4.2) the injective hulls $H(M_p)$ of the indecomposable submodules M_p are isomorphic to principal indecomposable submodules $\hat{P}_p$ of the left regular module ${}_{\hat{A}}\hat{A}$ [5]. Since the orthogonal idempotents $\{\hat{z}^R_p\}_p$ are central in $\hat{A}$ the principal indecomposable submodules $\hat{P}_p \subset {}_{\hat{A}}\hat{A}$ are also inequivalent for different p, therefore the injective hull $H(M)$ of M, hence M itself is isomorphic to a submodule of ${}_{\hat{A}}\hat{A}$. But for Frobenius algebras the isomorphism ${}_{\hat{A}}\hat{A} \simeq {}_{\hat{A}}A \equiv {}_{\hat{A}}(A, \rightharpoonup)$ holds [5], which implies that M is isomorphic to an invertible submodule of ${}_{\hat{A}}A$, hence, by iii) to a principal submodule $F_g \subset {}_{\hat{A}}A$ with $g \in G_R(A)$ ($g \in G_L(A)$). Due to ii) the inequivalent principal submodules can be characterized by the factor group $G_{R/L}/G^T_{R/L}$.

Now, it is enough to prove semisimplicity of the invertible $\hat{A}$-modules $F_g, g \in G_{R/L}(A)$. The right/left grouplike properties (5.1) of $g \in G_{R/L}(A)$ imply that $A_g := \hat{A} \rightharpoonup g \leftharpoonup \hat{A} = A^T g = g A^T$ is a subcoalgebra of A, and left (right) multiplication by an element g of $G_R(A)$ ($G_L(A)$) lead to $A^T \to A_g$ coalgebra isomorphism. Hence, not only A^T but also $A_g \subset A$ is a direct sum of simple coalgebras due to Lemma 2.3, i.e. A_g is contained in the coradical of C_0 of A. This implies that $F_g, g \in G_{R/L}(A)$ are completely reducible left $\hat{A}$-modules with respect to the left Sweedler action, because they are annihilated by the radical $\hat{N} = (C_0)^\perp$ of $\hat{A}$. Therefore the decomposition of an invertible $\hat{A}$-module in Lemma 5.3 ii) into (inequivalent) indecomposable submodules is a decomposition into (inequivalent) simple submodules, that is an invertible $\hat{A}$-module is semisimple.

Since a Frobenius algebra possesses finite number of inequivalent simple modules there is only a finite number of inequivalent modules that are direct sums of inequivalent simple modules. Therefore the factor groups $G_{R/L}(A)/G^T_{R/L}(A)$ are finite groups. ■

In consideration of Prop. 5.4 we can formulate why the notion of grouplike elements in a WHA is too restrictive: One cannot always associate a grouplike element in A to an invertible module of the dual WHA $\hat{A}$. We formulate this claim as follows:

Proposition 5.5 *Let $t_L \in A^L_*$ denote the element that relates the counit and the reduced trace as non-degenerate functionals on the separable algebra A^L: $\varepsilon(\cdot) = \mathrm{tr}\,(\cdot\, t_L)$. The coset $gG^T_R(A) \subset G_R(A)$ for $g \in G_R(A)$ contains a grouplike element iff there exists $x_L \in A^L_*$ such that*

$$gt_Lg^{-1} = x_Lt_Lx_L^{-1}. \tag{5.32}$$

In general, $G(A)/G^T(A)$ is a proper subgroup of $G_R(A)/G^T_R(A)$.

Proof: The adjoint action by $g \in G_R(A)$ on A gives rise to algebra automorphisms of A^L and A^R, because (5.1–2a) imply that $\Pi^{R/L}(gy_{R/L}g^{-1}) = gy_{R/L}g^{-1}$ for $y_{R/L} \in A^{R/L}$. Using the invariance of the reduced trace with respect to algebra automorphisms and the WBA identity $\varepsilon(abc) = \varepsilon(\Pi^R(a)b\Pi^L(c)); a, b, c \in A$, which follows from (1.1b) and (1.3) one obtains

$$\begin{aligned}\varepsilon(y_LgS(g)) &= \varepsilon(\Pi^R(g^{-1})y_L\Pi^L(g)) = \varepsilon(g^{-1}y_Lg) = \mathrm{tr}\,(g^{-1}y_Lgt_L)\\ &= \mathrm{tr}\,(y_Lgt_Lg^{-1}) = \varepsilon(y_Lgt_Lg^{-1}t_L^{-1}), \quad y_L \in A^L,\end{aligned} \tag{5.33}$$

i.e. $gS(g) = gt_Lg^{-1}t_L^{-1}$ due to non-degeneracy of the counit on A^L. Therefore for all $g \in G_R(A)$ we have

$$S(g) = t_Lg^{-1}t_L^{-1}, \qquad S^2(g) = tgt^{-1}, \quad t := t_LS(t_L^{-1}). \tag{5.34}$$

The element t_L implements the Nakayama automorphism $\theta_\varepsilon = S^2$ of ε on A^L, $\theta_\varepsilon = \mathrm{Ad}\, t_L$, hence $t := t_LS(t_L^{-1}) \in A^T$ implements S^2 on A^T and due to (5.34) on the subcoalgebras gA^T of A, $g \in G_R(A)$ as well. t is a grouplike element in the trivial subalgebra, $t \in G^T(A)$, because $t \in G^T_R(A)$ by Prop. 5.4 ii) and $S^2(t) = t$ also holds.

Hence, if for a given $g \in G_R(A)$ there exists $x_L \equiv x_L(g) \in A^L_*$ such that $gt_Lg^{-1} = x_Lt_Lx_L^{-1}$, then $gS(g) = gt_Lg^{-1}t_L^{-1} = x_Lt_Lx_L^{-1}t_L^{-1} = x_LS^2(x_L^{-1})$ due to (5.34) and $h := x_L^{-1}S(x_L)g \in G_R(A)$ is a grouplike element in the coset $gG^T_R(A)$ because $\Pi^L(h) = \mathbf{1}$.

Conversely, if h is a grouplike element in the coset $gG^T_R(A) \subset G_R(A)$ then $h = x_LS(x_L^{-1})g$ for some $x_L \in A^L_*$ due to Prop. 5.4 ii). Hence, using (5.34)

$$\mathbf{1} = \Pi^L(h) = x_LgS(g)S^2(x_L^{-1}) = x_Lgt_Lg^{-1}t_L^{-1}S^2(x_L^{-1}) = x_Lgt_Lg^{-1}x_L^{-1}t_L^{-1}. \tag{5.35}$$

For the second statement of the proposition first we note that the inclusion $gG^T(A) \subset gG^T_R(A)$ for $g \in G(A)$ induces the inclusion $G(A)/G^T(A) \subset G_R(A)/G^T_R(A)$ of the factor groups. To show that this inclusion is proper in general an example will suffice.

Let the WHA A over the rational field $\mathbf{Q}$ be given as follows. Let A^L be a full matrix algebra $M_n(\mathbf{Q}(\sqrt{2}))$, where $\mathbf{Q}(\sqrt{2})$ denotes the field extension of $\mathbf{Q}$ by $\sqrt{2}$. Let the counit as a non-degenerate index $\mathbf{1}$ functional on the separable algebra A^L be given by the help of the reduced trace and by $t_L \in A^L_*$ satisfying $\mathrm{tr}\,(t_L^{-1}) = 1$. Let A^T be the WHA of the form $A^L \otimes A^{Lop} \equiv A^L \otimes A^R$ given in the Appendix of [2]. Now let A as an algebra over $\mathbf{Q}$ be given by the crossed product $A := A^T \rtimes Z_2$, where $Z_2 = \{e, g\}$ is the cyclic group of order two and the action of the non-trivial element $g \in Z_2$ on A^L (A^R) is the outer automorphism that changes the sign of the central element $z_L = \sqrt{2}\cdot\mathbf{1}$ of A^L ($z_R = \sqrt{2}\cdot\mathbf{1} \in A^R$). Now it is a straightforward calculation that one extends the WHA structure of A^T to $A := A^T \rtimes Z_2$ by defining

$$\begin{aligned} \tilde{\varepsilon}(g^n x) &:= \epsilon(x), \\ \tilde{\Delta}(g^n x) &:= (g^n \otimes g^n)\Delta(x), \\ \tilde{S}(g^n x) &:= S(x) t_L g^n t_L^{-1}, \end{aligned} \tag{5.36}$$

where $x \in A^T$ and $n = 0, 1$.

It is also easy to check that $g \in A$ becomes a right grouplike element for any possible choice of t_L, that is $G_R(A)/G_R^T(A) \simeq Z_2$. However, if $t_L \in A_*^L$ is such that the prescribed outer automorphism on A^L induced by g is not inner on t_L, that is (5.32) does not fulfil, there is no grouplike element in the coset $gG_R^T(A) \subset G_R(A)$, hence $G(A)/G^T(A) \simeq \{e\}$. ■

Remark 5.6 We note that condition (5.32) in Prop. 5.5 fulfils by definition if A^L is central simple and it is also true if t_L is central in A^L, that is if $S^2_{|A^L} = id_{|A^L}$. Moreover, in the latter case the notions of various grouplike elements coincide: $G(A) = G_{R/L}(A)$. This happens, for example, if A^L is Abelian. Indeed, for $t_L \in \operatorname{Center} A^L$ the condition (5.32) $gt_Lg^{-1} = t_L, g \in G_R(A)$ follows, because due to (5.34) $S(g)g = t_L g^{-1} t_L^{-1} g$, and it is a central element in A^L. Therefore $\mathbf{1} = \Pi^R(g) = S(\mathbf{1}^{(1)})S(g)g\mathbf{1}^{(2)} = S(g)g$ due to (5.1–2a). ■

6. Distinguished (left/right) grouplike elements, Radford formula and the order of the antipode

After defining distinguished left/right grouplike elements and deriving some basic properties of them we prove that similarly to the finite dimensional Hopf case [15] the fourth power of the antipode in a WHA can be expressed by the help of distinguished left (right) grouplike elements. Using this result we derive a finiteness type claim about the order of the antipode in a WHA an prove that the double of a WHA is unimodular.

We note that the Radford formula was proved in [13] for WHAs in the special case when the square of the antipode is the identity mapping on A^L.[1] For such a special WHA A the notions of various grouplike elements coincide, $G(A) = G_{R/L}(A)$, due to Remark 5.6.

Before turning to the definition of (left/right) distinguished grouplike elements in a WHA let us examine the connection between integrals in dual pairs $A, \hat{A}$ of WHAs.

[1] For WHAs based on certain separable, but not strongly separable [9] algebra A^L the property $S^2_{|A^L} \neq id_{|A^L}$, i.e. the non-triviality of the Nakayama automorphism corresponding to the counit as a non-degenerate functional $\varepsilon\colon A^L \to k$, is not only a possibility, but the only possibility, because ε should be an index $\mathbf{1}$ functional on A^L. For example, if $A^L = M_2(\mathbf{Z}_2)$, that is a two by two matrix algebra over the finite field $\mathbf{Z}_2$, the reduced trace tr on A^L is non-degenerate but it has index 0. The two non-degenerate index $\mathbf{1}$ functional on A^L have the form $\operatorname{tr}(\cdot\, t_L)$ with $t_L^{\pm 1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and lead to $S^2_{|A^L} = \operatorname{Ad} t_L \neq id_{|A^L}$.

The pair $(l,\lambda), l \in I^L \subset A, \lambda \in \hat{I}^L \subset \hat{A}$ is called a *dual pair of left integrals* if they are non-degenerate and if they obey one of the equivalent relations $l \rightharpoonup \lambda = \hat{\mathbf{1}}, \lambda \rightharpoonup l = \mathbf{1}$. Due to Theorem 4.1 such pairs exist in any dual pair $A, \hat{A}$ of WHAs. In the necessity proof of Theorem 4.1 we have also seen that the left A-module $({}_A\hat{I}^L, \star)$ becomes a free left $A^{L/R}$-module with a single generator by restricting A to the subalgebras $A^{L/R}$. Hence, $({}_A\hat{I}^L, \star)$ is an invertible A-module due to Lemma 5.3 i). Since this module is the right conjugate of the module ${}_AI^R$ due to Corollary 3.4 ${}_AI^R$ is also an invertible left A-module and it becomes a free left $A^{L/R}$-module with a single generator by restricting A to the subalgebras $A^{L/R}$ by Lemma 5.3 i). An element r is a free A^L (A^R) generator in ${}_AI^R$ iff r is a non-degenerate right integral, thus non-degenerate right integrals $r, r' \in I^R$ are related by an invertible element $x_L \in A^L$ ($x_R \in A^R$): $r' = x_L r$ ($r' = x_R r$). By duality the corresponding statement holds for non-degenerate right integrals in $\hat{I}^R$. Hence *dual pairs of right integrals* (r_1, ρ_1) and (r_2, ρ_2), i.e when $r_i \in I^R, \rho_i \in \hat{I}^R$ are non-degenerate and obey one of the equivalent relations $r_i \leftharpoonup \rho_i = \mathbf{1}, \rho_i \leftharpoonup r_i = \hat{\mathbf{1}}; i = 1, 2$, are related by a 'common' invertible element $x_L \in A^L$ ($x_R \in A^R$):

$$(r_2, \rho_2) = (x_L r_1, (\hat{\mathbf{1}} \leftharpoonup x_L^{-1})\rho_1) = (x_R r_1, (S^{-2}(x_R^{-1}) \rightharpoonup \hat{\mathbf{1}})\rho_1). \tag{6.1}$$

Now, let us consider the element $\rho \rightharpoonup r \in A$ constructed from the elements of the dual pair (r, ρ) of right integrals. Since r is a non-degenerate functional on $\hat{A}$ and since ρ is a free left $\hat{A}^{L/R}$-generator of the left $\hat{A}$-module ${}_{\hat{A}}\hat{I}^R$ by restricting $\hat{A}$ to these subalgebras, $\rho \rightharpoonup r$ becomes a free left $\hat{A}^{L/R}$-generator of the left $\hat{A}$-module ${}_{\hat{A}}(\hat{A} \rightharpoonup (\rho \rightharpoonup r), \rightharpoonup)$, i.e. it is an invertible submodule in ${}_{\hat{A}}(A, \rightharpoonup)$. Moreover,

$$\begin{aligned}\Pi^R(\rho \rightharpoonup r) &= \Pi^R(r^{(1)})\langle r^{(2)}, \rho\rangle = \mathbf{1}^{(1)}\langle r\mathbf{1}^{(2)}, \rho\rangle \\ &= \mathbf{1}^{(1)}\langle \mathbf{1}^{(2)}, \rho \leftharpoonup r\rangle = \mathbf{1}^{(1)}\langle \mathbf{1}^{(2)}, \hat{\mathbf{1}}\rangle = \mathbf{1},\end{aligned} \tag{6.2}$$

that is $\rho \rightharpoonup r \in A$ is a right grouplike element due to Prop. 5.4 i). If $(r_i, \rho_i); i = 1, 2$ are dual pairs of right integrals the corresponding right grouplike elements differ by a right grouplike element in A^T due to (6.1) and Prop 5.4 ii):

$$\rho_2 \rightharpoonup r_2 = (\hat{\mathbf{1}} \leftharpoonup x_L^{-1})\rho_1 \rightharpoonup x_L r_1 = x_L S(x_L^{-1})(\rho_1 \rightharpoonup r_1), \qquad x_L \in A_*^L. \tag{6.3}$$

However, it is not known to us whether the coset $G_R^T(A)(\rho \rightarrow r)$ in $G_R(A)$ is special enough in order to contain always a grouplike element. But we note that if for a dual pair (r, ρ) of right integrals $\rho \rightharpoonup r$ is grouplike, i.e. $\Pi^L(\rho \rightharpoonup r) = \mathbf{1}$ due to Prop. 5.4 i), then $r \rightharpoonup \rho \in G(\hat{A})$ already follows: By duality $r \rightharpoonup \rho$ is a free $A^{L/R}$-generator in the left A-module ${}_A(A \rightharpoonup (r \rightharpoonup \rho), \rightharpoonup)$ with $\hat{\Pi}^R(r \rightharpoonup \rho) = \hat{\mathbf{1}}$ and

$$\begin{aligned}\hat{\Pi}^L(r \rightharpoonup \rho) &= \hat{\Pi}^L(\rho^{(1)})\langle \rho^{(2)}, r\rangle = \hat{S}(\hat{\mathbf{1}}^{(1)})\langle \hat{\mathbf{1}}^{(2)}\rho, r\rangle \\ &= \hat{S}(\Pi^L(\rho \rightharpoonup r) \rightharpoonup \hat{\mathbf{1}}) = \hat{S}(\mathbf{1} \rightharpoonup \hat{\mathbf{1}}) = \hat{\mathbf{1}},\end{aligned} \tag{6.4}$$

that is $r \rightharpoonup \rho$ is grouplike by Prop. 5.4 i).

Similarly, one can show that a dual pair (l, λ) of left integrals leads to left grouplike elements: $l \leftharpoonup \lambda \in G_L(A)$ and $\lambda \leftharpoonup l \in G_L(\hat{A})$. These considerations lead to the following

Definition 6.1 *Let $(l,\lambda)/(r,\rho)$ be dual pair of left/right integrals in a dual pair $A,\hat{A}$ of WHAs. The elements $s_L := l \leftharpoonup \lambda$ ($\sigma_L := \lambda \leftharpoonup l$) and $s_R := \rho \rightharpoonup r$ ($\sigma_R := r \rightharpoonup \rho$) are called distinguished left and right grouplike elements, respectively, in A ($\hat{A}$).*
The dual pair $(l,\lambda)/(r,\rho)$ of left/right integrals is called a distinguished pair of left/right integrals if s_L/s_R is not only left/right grouplike but also grouplike. In this case s_L and s_R are called distinguished grouplike elements.

Remark 6.2 The relation between the Π^L projections of distinguished right grouplike elements σ_R and s_R is given by (6.4). Similarly, for distinguished left grouplike elements one obtains the relation

$$\Pi^R(s_L) = S(\mathbf{1} \leftharpoonup \hat{\Pi}^R(\sigma_L)) = S(\mathbf{1} \leftharpoonup \sigma_L). \tag{6.5}$$

Since $(S(r), \hat{S}^{-1}(\rho))$ is a dual pair of left integrals if (r,ρ) is a dual pair of right integrals the cosets of left and right grouplike elements s_L and s_R are related as $s_L G_L^T(A) = S(s_R)G_L^T(A) = t_L s_R^{-1} t_L^{-1} G_L^T(A)$ due to (5.34). ■

Let us introduce some notations we use in the forthcoming Lemma: Using properties (5.1–2) it is easy to see that a left/right grouplike element $\gamma_{L/R} \in G_{L/R}(\hat{A})$ gives rise to a projection $\Pi^{L/R}_{\gamma_{L/R}}\colon A \to A^{L/R}$ by defining

$$\Pi^L_{\gamma_L}(a) := \Pi^L(\gamma_L \rightharpoonup a), \qquad \Pi^R_{\gamma_R}(a) := \Pi^R(a \leftharpoonup \gamma_R), \quad a \in A. \tag{6.6}$$

The invertible right/left A-module structures of left/right integrals in A can be made explicit by using these projections and distinguished left/right grouplike elements $\sigma_{L/R}$ connected to the dual pair $(l,\lambda)/(r,\rho)$ of left/right integrals:

$$la = l\Pi^R_{\hat{S}(\sigma_L^{-1})}(a), \qquad ar = \Pi^L_{\hat{S}(\sigma_R^{-1})}(a)r, \quad a \in A. \tag{6.7}$$

For example the first relation can be proved by using (5.1–2b) and the injectivity of $\hat{L}_\lambda$:

$$\begin{aligned}\lambda \leftharpoonup la &= \sigma_L \leftharpoonup a = \langle \hat{S}(\sigma_L^{-1})\hat{\mathbf{1}}^{(1)}, a\rangle \sigma_L \hat{\mathbf{1}}^{(2)} = \sigma_L(\hat{\mathbf{1}} \leftharpoonup \Pi^R(a \leftharpoonup \hat{S}(\sigma_L^{-1})))\\ &= \sigma_L \leftharpoonup \Pi^R(a \leftharpoonup \hat{S}(\sigma_L^{-1})) = \lambda \leftharpoonup l\Pi^R_{\hat{S}(\sigma_L^{-1})}(a).\end{aligned}$$

Lemma 6.3 *Let $P_b = AbA \subset A$ be the principal ideal with the generator $b = b(\gamma,\delta) \in A$ characterized by a left and a right grouplike element $\gamma \in G_L(\hat{A}), \delta \in G_R(\hat{A})$ through the property*

$$abc = \Pi^L_\gamma(a)b\Pi^R_\delta(c), \qquad a, c \in A, \tag{6.8}$$

where the projections Π^L_γ and Π^R_δ are defined in (6.6). The left/right Sweedler action by left/right grouplike elements provide isomorphisms between such types of principal ideals as (possibly non-unital) rings. The image $\tilde{b}$ of the generator $b = b(\gamma,\delta)$ obeys the characterization property

$$\beta_L \rightharpoonup b \leftharpoonup \beta_R \equiv \tilde{b} = \tilde{b}(\hat{S}(\beta_R)\gamma\beta_L^{-1}, \beta_R^{-1}\delta\hat{S}(\beta_L)), \qquad \beta_{L/R} \in G_{L/R}(\hat{A}). \tag{6.9}$$

Proof. First we note that the set of such principal ideals is non-empty, for example, $l \in I^L$ from a dual pair (l, λ) of left integrals provides such a generator: $l = l(\hat{\mathbf{1}}, \hat{S}(\sigma_L^{-1}))$ due to (6.7), where $\sigma_L = l \leftharpoondown \lambda \in G_L(\hat{A})$ is the corresponding distinguished left grouplike element.

Since left/right Sweedler actions by left/right grouplike elements in $\hat{A}$ provide algebra automorphisms of A the isomorphism of the corresponding principal ideals as rings follows. The only open question is the transformation property (6.9) of the generator $b = b(\gamma, \delta)$. Using defining properties (5.1–2) of left/right grouplike elements and the properties of the maps κ_R, κ_L given in (1.5–6)

$$\begin{aligned} a(\beta_L \rightharpoonup b) &= \beta_L \rightharpoonup (\beta_L^{-1} \rightharpoonup a)b = \beta_L \rightharpoonup \Pi_\gamma^L(\beta_L^{-1} \rightharpoonup a)b = \Pi_\gamma^L(\beta_L^{-1} \rightharpoonup a)(\beta_L \rightharpoonup b) \\ &= \Pi^L(\gamma \rightharpoonup \beta_L^{-1} \rightharpoonup a)(\beta_L \rightharpoonup b) = \Pi^L_{\gamma\beta_L^{-1}}(a)(\beta_L \rightharpoonup b), \end{aligned} \tag{6.10a}$$

$$\begin{aligned} (\beta_L \rightharpoonup b)c &= \beta_L \rightharpoonup b(\beta_L^{-1} \rightharpoonup c) = \beta_L \rightharpoonup b\Pi_\delta^R(\beta_L^{-1} \rightharpoonup c) \\ &= (\beta_L \rightharpoonup b)(\beta_L \rightharpoonup \Pi_\delta^R(\beta_L^{-1} \rightharpoonup c)) = (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\langle \beta_L, \Pi_\delta^R(\beta_L^{-1} \rightharpoonup c)\mathbf{1}^{(2)}\rangle \\ &= (\beta_L \rightharpoonup b)((\beta_L \leftharpoondown \Pi_\delta^R(\beta_L^{-1} \rightharpoonup c)) \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)(\hat{\Pi}^L(\beta_L(\hat{\mathbf{1}} \leftharpoondown \Pi^R(\beta_L^{-1} \rightharpoonup c \leftharpoondown \delta))) \rightharpoonup \mathbf{1} \\ &= (\beta_L \rightharpoonup b)(\hat{\Pi}^L(\beta_L(\hat{\mathbf{1}} \leftharpoondown (\beta_L^{-1} \rightharpoonup c \leftharpoondown \delta)) \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)\langle \hat{\mathbf{1}}^{(1)}, \beta_L^{-1} \rightharpoonup c \leftharpoondown \delta\rangle(\beta_L^{(1)}\hat{\mathbf{1}}^{(2)}\hat{S}(\beta_L^{(2)}) \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)\langle \hat{\mathbf{1}}^{(1)}, \beta_L^{-1} \rightharpoonup c \leftharpoondown \delta\rangle(\beta_L\mathbf{1}^{(2)}\beta_L^{-1} \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)\langle \hat{S}(\beta_L)\hat{\mathbf{1}}^{(1)}\beta_L, \beta_L^{-1} \rightharpoonup c \leftharpoondown \delta\rangle(\hat{\mathbf{1}}^{(2)} \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)\langle \hat{\mathbf{1}}^{(1)}, c \leftharpoondown \delta\hat{S}(\beta_L)\rangle(\hat{\mathbf{1}}^{(2)} \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)((\hat{\mathbf{1}} \leftharpoondown \Pi^R(c \leftharpoondown \delta\hat{S}(\beta_L))) \rightharpoonup \mathbf{1}) \\ &= (\beta_L \rightharpoonup b)\Pi^R(c \leftharpoondown \delta\hat{S}(\beta_L)) = (\beta_L \rightharpoonup b)\Pi^R_{\delta\hat{S}(\beta_L)}(c). \end{aligned} \tag{6.10b}$$

The transformation property (6.9) of the characterizing left/right grouplike elements under the right Sweedler action $b \leftharpoondown \beta_R, \beta_R \in G_R(\hat{A})$ can be proved similarly. ■

Corollary 6.4 *Distinguished left grouplike elements in $\hat{A}$ fall into central elements of the factor group $G_L(\hat{A})/G_L^T(\hat{A})$. There exists a two-sided non-degenerate integral in A iff distinguished left grouplike elements in $\hat{A}$ fall into the unit element of this factor group.*

Proof. For any $\beta \in G_L(\hat{A})$ the map $B_\beta(a) := \beta \rightharpoonup a \leftharpoondown \hat{S}^{-1}(\beta), a \in A$ defines an algebra automorphism of A, which maps the space I^L of left integrals into itself due to the previous Lemma. The image $\tilde{l} \equiv B_\beta(l)$ of a non-degenerate left integral $l = l(\hat{\mathbf{1}}, \hat{S}(\sigma_L^{-1}))$ is a non-degenerate left integral having the characterization property $\tilde{l} = \tilde{l}(\hat{\mathbf{1}}, \hat{S}^{-1}(\beta^{-1})\hat{S}(\sigma_L^{-1})\hat{S}(\beta))$ due to (6.9). Hence, the distinguished left grouplike element $\tilde{\sigma}_L$ corresponding to $\tilde{l}$ is given by

$$\tilde{\sigma}_L = \hat{S}^{-2}(\beta)\sigma_L\beta^{-1} = \varphi\beta\sigma_L\beta^{-1}, \quad S^{-2}(\beta)\beta^{-1} =: \varphi = \hat{S}^{-1}(\varphi_L^{-1})\varphi_L \in G_L^T(\hat{A}), \tag{6.11}$$

with $\varphi_L = \hat{S}^{-1}(\hat{\Pi}^R(\beta^{-1})) \in \hat{A}_*^L$ due to the form (5.2b) of $\hat{\Pi}^R(\beta^{-1})$. However, distinguished left grouplike elements, similarly to the case (6.3) of distinguished right grouplike elements, differ by elements in $G_L^T(\hat{A})$, hence for the cosets (6.11) implies $[\sigma_L] = [\varphi][\beta][\sigma_L][\beta]^{-1} = [\beta][\sigma_L][\beta]^{-1}$, that is $[\sigma_L]$ is central in the factor group $G_L(\hat{A})/G_L^T(\hat{A})$.

If the non-degenerate left integral $l \in I^L$ is also a right integral then we have the relation $\Pi^R_{\hat{S}(\sigma_L^{-1})} = \Pi^R$ due to (6.7). Hence, $\sigma_L = \hat{\mathbf{1}}$ since

$$\langle \hat{\mathbf{1}}, a\rangle = \langle \hat{\mathbf{1}}, \Pi^R(a)\rangle = \langle \hat{\mathbf{1}}, \Pi^R_{\hat{S}(\sigma_L^{-1})}(a)\rangle = \langle \hat{S}(\sigma_L^{-1}) \otimes \hat{\mathbf{1}}, \Delta(a)\rangle = \langle \hat{S}(\sigma_L^{-1}), a\rangle, \quad a \in A.$$

Conversely, if $[\sigma_L]$ is the unit element of the factor group then there exists a dual pair (l, λ) of left integrals with distinguished left grouplike element $\sigma_L = \hat{\mathbf{1}}$. Then $\Pi^R_{\hat{S}(\sigma_L^{-1})} = \Pi^R$ and (6.7) implies that l is a (non-degenerate) two-sided integral. ■

Theorem 6.5 *Let $A, \hat{A}$ be a dual pair of WHAa. If (l, λ) is a dual pair of left integrals and (s_L, σ_L) is the corresponding pair of distinguished left grouplike elements in $A \times \hat{A}$ then the Nakayama automorphism $\theta_\lambda := \hat{R}_\lambda^{-1} \circ \hat{L}_\lambda \colon A \to A$ corresponding to the non-degenerate functional $\lambda \colon A \to k$ can be written as*

$$\theta_\lambda(a) = \sigma_L^{-1} \rightharpoonup S^2(a) = s_L^{-1} S^{-2}(a) s_L \leftharpoonup \hat{S}^{-1}(\sigma_L), \qquad a \in A. \tag{6.12}$$

The fourth power of the antipode S of A can be written as:

$$S^4(a) = \sigma_L \rightharpoonup s_L^{-1} a s_L \leftharpoonup \hat{S}^{-1}(\sigma_L), \qquad a \in A. \tag{6.13}$$

The order of the antipode is finite up to an inner automorphism by a grouplike element in the trivial subalgebra A^T.

Proof. In the sufficiency proof of Theorem 4.1 we have seen that the antipode and its inverse can be given by the help of pairs of non-degenerate integrals $l/r \in I^{L/R}, \lambda/\rho \in \hat{I}^{L/R}$

$$S(a) = (R_l \circ \hat{L}_\lambda)(a) := (\lambda \leftharpoonup a) \rightharpoonup l, \qquad \lambda \rightharpoonup l = \mathbf{1}, l \rightharpoonup \lambda = \hat{\mathbf{1}}, \tag{6.14a}$$
$$S^{-1}(a) = (L_l \circ \hat{L}_\rho)(a) := l \leftharpoonup (\rho \leftharpoonup a), \qquad l \leftharpoonup \rho = \mathbf{1}, l \rightharpoonup \rho = \hat{\mathbf{1}} \tag{6.14b}$$
$$S^{-1}(a) = (R_r \circ \hat{R}_\lambda)(a) := (a \rightharpoonup \lambda) \rightharpoonup r, \qquad \lambda \leftharpoonup r = \hat{\mathbf{1}}, \lambda \rightharpoonup r = \mathbf{1}, \tag{6.14c}$$
$$S(a) = (L_r \circ \hat{R}_\rho)(a) := r \leftharpoonup (a \rightharpoonup \rho), \qquad r \leftharpoonup \rho = \mathbf{1}, \rho \leftharpoonup r = \hat{\mathbf{1}}. \tag{6.14d}$$

Choosing a dual pair (l, λ) of left integrals we rewrite the antipode relations (6.14b–d) in terms of $(l.\lambda)$ and the corresponding pair $(s, \sigma) \equiv (s_L, \sigma_L)$ of distinguished left grouplike elements. We note that the second relations between the members of integral pairs given in (6.14a–d) are consequences of the first ones (see the proof of Theorem 4.1), hence, it is enough to ensure only these ones.

For (6.14b) the new member of the required pair (l, ρ) of integrals is given by $\rho := \hat{S}^{-1}(\lambda) = (\lambda \leftharpoonup s)\hat{\Pi}^R(\sigma)^{-1}$. Indeed, ρ is a non-degenerate right integral and $\lambda = \hat{S}(\rho) = (l \leftharpoonup \rho) \rightharpoonup \lambda$ implies the relation $l \leftharpoonup \rho = \mathbf{1}$ due to injectivity of $\hat{R}_\lambda$. Moreover, using property (1.16) of left integrals

$$\begin{aligned}(\lambda \leftharpoonup s)\hat{\Pi}^R(\sigma)^{-1} &:= (\lambda \leftharpoonup (l \leftharpoonup \lambda))\hat{\Pi}^R(\sigma)^{-1} = \langle \lambda\lambda^{(1)}, l\rangle \lambda^{(2)}\hat{\Pi}^R(\sigma)^{-1} \\ &= \langle \lambda^{(1)}, l\rangle \hat{S}^{-1}(\lambda)\lambda^{(2)}\hat{\Pi}^R(\sigma)^{-1} = \hat{S}^{-1}(\lambda)\sigma\hat{\Pi}^R(\sigma)^{-1} \\ &= \hat{S}^{-1}(\lambda)\hat{\Pi}^R(\sigma)\hat{\Pi}^R(\sigma)^{-1} = \hat{S}^{-1}(\lambda) = \rho.\end{aligned} \tag{6.15}$$

Hence, interchanging the role of A and $\hat{A}$ the new member of integrals for (6.14c) is given by $r := S^{-1}(l) = (l \leftharpoonup \sigma)\Pi^R(s)^{-1}$. For (6.14d) the pair is given by $(r := S^{-1}(l), \rho := \hat{S}(\lambda) = s \rightharpoonup \lambda)$, because $\rho = \hat{S}(\lambda) = (l \leftharpoonup \lambda) \rightharpoonup \lambda = s \rightharpoonup \lambda$ and $r = S^{-1}(l)$ are non-degenerate right integrals and $r \leftharpoonup \rho = S^{-1}(l) \leftharpoonup \hat{S}(\lambda) = S^{-1}(\lambda \rightharpoonup l) = \mathbf{1}$. Therefore we can rewrite (6.14b–c) as

$$\begin{aligned} S^{-1}(a) &= l \leftharpoonup (\rho \leftharpoonup a) = l \leftharpoonup ((\lambda \leftharpoonup s)\hat{\Pi}^R(\sigma)^{-1} \leftharpoonup a) = l \leftharpoonup (\lambda \leftharpoonup sa)\hat{\Pi}^R(\sigma)^{-1} \\ &= [l \leftharpoonup (\lambda \leftharpoonup sa)](\mathbf{1} \leftharpoonup \hat{\Pi}^R(\sigma)^{-1}) = (L_l \circ \hat{L}_\lambda)(sa)(\mathbf{1} \leftharpoonup \hat{\Pi}^R(\sigma)^{-1}), \qquad (6.16b) \\ S^{-1}(\sigma \rightharpoonup a) &= S^{-1}(a) \leftharpoonup \hat{S}(\sigma) = ((a \rightharpoonup \lambda) \rightharpoonup r) \leftharpoonup \hat{S}(\sigma) \\ &= (a \rightharpoonup \lambda) \rightharpoonup (l \leftharpoonup \sigma)\Pi^R(s)^{-1} \leftharpoonup \hat{S}(\sigma) = (a \rightharpoonup \lambda) \rightharpoonup (l \leftharpoonup \sigma\hat{S}(\sigma))\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup (l \leftharpoonup \hat{\Pi}^R(\sigma^{-1})^{-1})\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l(\mathbf{1} \leftharpoonup \hat{\Pi}^R(\sigma^{-1})^{-1})\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l\Pi^R((\mathbf{1} \leftharpoonup \sigma\hat{S}(\sigma)) \leftharpoonup \hat{S}(\sigma^{-1}))\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup lS(\mathbf{1} \leftharpoonup \sigma)\Pi^R(s)^{-1} = (a \rightharpoonup \lambda) \rightharpoonup l\Pi^R(s)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l = (R_l \circ \hat{R}_\lambda)(a), \qquad (6.16c) \end{aligned}$$

where we used the identity $\sigma\hat{S}(\sigma) = \hat{\Pi}^R(\sigma^{-1})^{-1}$ following from (5.2b), the right A-module property (6.7) of left integrals and the relation (6.5). Finally, using property (1.16) of left integrals (6.14d) can be rewritten as

$$\begin{aligned} S(a) &= r \leftharpoonup (a \rightharpoonup \rho) = (l \leftharpoonup \sigma)\Pi^R(s)^{-1} \leftharpoonup (a \rightharpoonup (s \rightharpoonup \lambda)) \\ &= [(l \leftharpoonup \sigma) \leftharpoonup (as \rightharpoonup \lambda)]\Pi^R(s)^{-1} = [l \leftharpoonup \sigma\lambda^{(1)}\langle\lambda^{(2)}, as\rangle]\Pi^R(s)^{-1} \\ &= [l \leftharpoonup \lambda^{(1)}\langle\hat{S}^{-1}(\sigma)\lambda^{(2)}, as\rangle]\Pi^R(s)^{-1} = [l \leftharpoonup ((as \leftharpoonup \hat{S}^{-1}(\sigma)) \rightharpoonup \lambda)]\Pi^R(s)^{-1} \\ &= (L_l \circ \hat{R}_\lambda)(as \leftharpoonup \hat{S}^{-1}(\sigma))\Pi^R(s)^{-1}. \qquad (6.16d) \end{aligned}$$

Therefore using (6.14a), (6.16b–d), the algebra isomorphism property of the map $\hat{\kappa}_R$ given in (1.5), the relation (6.5) and the form (5.2b) of $\Pi^R(s)$ we get

$$\begin{aligned} (R_l \circ \hat{L}_\lambda)(a) &= S(a) = S^{-1}(\sigma \rightharpoonup (\sigma^{-1} \rightharpoonup S^2(a))) = (R_l \circ \hat{R}_\lambda)(\sigma^{-1} \rightharpoonup S^2(a)), \quad (6.17a) \\ (L_l \circ \hat{L}_\lambda)(a) &= S^{-1}(s^{-1}a)(\mathbf{1} \leftharpoonup \hat{\Pi}^R(\sigma)) = S^{-1}(\Pi^R(s)s^{-1}a) \\ &= S[S^{-1}(\Pi^R(s)^{-1})S^{-2}(\Pi^R(s)s^{-1}a)]\Pi^R(s) \\ &= S[S^{-2}(S(\Pi^R(s)^{-1})\Pi^R(s)s^{-1}a)]\Pi^R(s) = S[S^{-2}(S^2(s^{-1})a)]\Pi^R(s) \\ &= S[s^{-1}S^{-2}(a)]\Pi^R(s) = (L_l \circ \hat{R}_\lambda)(s^{-1}S^{-2}(a)s \leftharpoonup \hat{S}^{-1}(\sigma)). \qquad (6.17b) \end{aligned}$$

Due to injectivity of R_l and L_l (6.17a–b) lead to connections between $\hat{R}_\lambda$ and $\hat{L}_\lambda$ that imply (6.12). The equality of these two different forms of the Nakayama automorphism θ_λ gives rise to the Radford formula (6.13).

Since left/right Sweedler actions by left/right grouplike elements are algebra automorphisms iterating the Radford formula m times one arrives at

$$S^{4m}(a) = S^{4m}(s^{-1})\dots S^4(s^{-1})(\sigma^m \rightharpoonup a \leftharpoonup \hat{S}^{-1}(\sigma^m))S^4(s)\dots S^{4m}(s), \qquad a \in A. \quad (6.18)$$

For $g \in G_L(A)$ the relation $S^2(g) = S(\Pi^R(g^{-1})^{-1})\Pi^R(g^{-1})g \in G_L^T(A)g$ holds due to (5.2b) and Prop. 5.4 ii), hence, $S^{2n}(g) \in G_L^T(A)g$ is also true for any integer n. However, the factor group $G_L(A)/G_L^T(A)$ is finite due to Prop. 5.4 iv) therefore there exists an integer m and $x \equiv S(x_R)x_R^{-1} \in G_L^T(A), \varphi \equiv \hat{S}(\varphi_R)\varphi_R^{-1} \in G_L^T(\hat{A})$ with $x_R \in A_*^R, \varphi_R \in \hat{A}_*^R$ such that (6.18) reads as

$$\begin{aligned} S^{4m}(a) &= x^{-1}(\varphi \rightharpoonup a \leftharpoonup \hat{S}^{-1}(\varphi))x \\ &= x^{-1}(\hat{S}(\varphi_R) \rightharpoonup \mathbf{1})(\mathbf{1} \leftharpoonup \hat{S}^{-1}(\varphi_R^{-1}))a(\varphi_R^{-1} \rightharpoonup \mathbf{1})(\mathbf{1} \leftharpoonup \varphi_R)x \\ &= x^{-1}S^{-1}(\mathbf{1} \leftharpoonup \varphi_R)(\mathbf{1} \leftharpoonup \varphi_R^{-1})aS^{-1}(\mathbf{1} \leftharpoonup \varphi_R^{-1})(\mathbf{1} \leftharpoonup \varphi_R)x \\ &= x^{-1}S^{-1}(\mathbf{1} \leftharpoonup \varphi_R)(\mathbf{1} \leftharpoonup \varphi_R)^{-1}aS^{-1}(\mathbf{1} \leftharpoonup \varphi_R)^{-1}(\mathbf{1} \leftharpoonup \varphi_R)x \\ &= S(y_R^{-1})y_R a S(y_R)y_R^{-1}, \qquad a \in A \end{aligned} \tag{6.19}$$

with $y_R = x_R S^{-1}(\mathbf{1} \leftharpoonup \varphi_R) \in A_*^R$, that is S^{4m} is an inner algebra automorphism of A by an element $y := S(y_R)y_R^{-1} \in G_L^T(A)$. However, S^{4m} is also a coalgebra automorphism of A, which requires y to be a grouplike element. Indeed, using the coproduct property (1.4) and separability identities (1.12) for A^L and A^R one derives the relation

$$\begin{aligned} \Delta(a) &= (S(y_R) \otimes y_R^{-1})\Delta(S(y_R^{-1})y_R a S(y_R)y_R^{-1})(S(y_R^{-1}) \otimes y_R) \\ &= (S(y_R) \otimes y_R^{-1})\Delta(S^{4m}(a))(S(y_R^{-1}) \otimes y_R) \\ &= (S(y_R) \otimes y_R^{-1})(S^{4m} \otimes S^{4m})(\Delta(a))(S(y_R^{-1}) \otimes y_R) \\ &= (y_R \otimes S(y_R^{-1}))\Delta(a)(y_R^{-1} \otimes S(y_R)) = (y_R S^2(y_R^{-1}) \otimes \mathbf{1})\Delta(a)(y_R y_R^{-1} \otimes \mathbf{1}) \\ &= (y_R S^2(y_R^{-1}) \otimes \mathbf{1})\Delta(a), \qquad a \in A, \end{aligned} \tag{6.20}$$

that leads to the equality $\mathbf{1} = y_R S^2(y_R^{-1})$ by applying the counit to the second tensor factor. Hence, $y = S(y_R)y_R^{-1}$ is not only in $G_L^T(A)$ but also in $G^T(A)$ due to Prop. 5.4 ii), which together with (6.19) proves the last claim in the theorem. ■

In case of Hopf algebras the original result [15] was used to prove that the Drinfeld double $\mathcal{D}(H)$ of a Hopf algebra H is unimodular [16]. In case of the double $\mathcal{D}(A)$ of a WHA A [3] the same result holds:

Corollary 6.6 *The double $\mathcal{D}(A)$ of a WHA A is unimodular, i.e. there exists a non-degenerate two-sided integral in $\mathcal{D}(A)$. Namely, if (l, λ) is a dual pair of left integrals in $A \times \hat{A}$ then $\mathcal{D}(l \otimes \hat{S}(\lambda))$ is a two-sided non-degenerate integral in $\mathcal{D}(A)$.*

Proof. The double $\mathcal{D}(A)$ of a WHA A is the tensor product of A and $\hat{A}$ amalgamated by $A^L \simeq \hat{A}^R$ and $A^R \simeq \hat{A}^L$

$$\mathcal{D}(A) \ni \mathcal{D}(ax_L x_R \otimes \varphi) = \mathcal{D}(a \otimes (x_L \rightharpoonup \hat{\mathbf{1}})(\hat{\mathbf{1}} \leftharpoonup x_R)\varphi) \tag{6.21}$$

with $a \in A, \varphi \in \hat{A}, x_{L/R} \in A^{L/R}$ together with the WHA structure maps

$$\begin{aligned} \mathcal{D}(a \otimes \varphi)\mathcal{D}(b \otimes \psi) &:= \mathcal{D}(ab^{(2)} \otimes \varphi^{(2)}\psi)\langle a^{(1)}, \hat{S}^{-1}(\varphi^{(3)})\rangle\langle a^{(3)}, \varphi^{(1)}\rangle, \\ \varepsilon_{\mathcal{D}}(\mathcal{D}(a \otimes \varphi)) &:= \varepsilon(a(\varphi \rightharpoonup \mathbf{1})) = \hat{\varepsilon}((\hat{\mathbf{1}} \leftharpoonup a)\varphi), \\ \Delta_{\mathcal{D}}(\mathcal{D}(a \otimes \varphi)) &:= \mathcal{D}(a^{(1)} \otimes \varphi^{(2)}) \otimes \mathcal{D}(a^{(2)} \otimes \varphi^{(1)}), \\ S_{\mathcal{D}}(\mathcal{D}(a \otimes \varphi)) &:= \mathcal{D}(\mathbf{1} \otimes \hat{S}^{-1}(\varphi))\mathcal{D}(S(a) \otimes \hat{\mathbf{1}}). \end{aligned} \tag{6.22}$$

Let (l,λ) be a dual pair of left integrals in $A\times\hat{A}$ with the corresponding pair $(s,\sigma)\equiv(s_L,\sigma_L)$ of distinguished left grouplike elements. From the form (6.12) of the Nakayama automorphism θ_l corresponding to l it follows that $\Delta^{op}(l)=l^{(1)}\otimes S^2(l^{(2)}s_L^{-1})$, hence using properties (5.1–2b) of a left grouplike element

$$\begin{aligned}l^{(2)}\otimes l^{(3)}s^{-1}S^{-1}(l^{(1)}) &= l^{(1)}\otimes l^{(2)}s^{-1}S(l^{(3)}s^{-1}) = l^{(1)}\otimes l^{(2)}\mathbf{1}^{(1)}s^{-1}S(s^{-1})S(\mathbf{1}^{(2)})S(l^{(3)})\\ &= l^{(1)}\otimes l^{(2)}\Pi^L(s^{-1})S(l^{(3)}) = l^{(1)}\otimes\Pi^L(l^{(2)}).\end{aligned}\tag{6.23}$$

Moreover, the first relation in (6.7) for $\lambda\in\hat{I}^L$ and $\varphi\in\hat{A}$ implies that

$$\varphi\hat{S}(\lambda)=\hat{S}(\lambda\hat{\Pi}^R_{S(s^{-1})}(\hat{S}^{-1}(\varphi)))=(\hat{S}\circ\hat{\Pi}^R\circ\hat{S}^{-1})(s^{-1}\rightharpoonup\varphi)=\hat{\Pi}^L_{s^{-1}}(\varphi)\hat{S}(\lambda).\tag{6.24}$$

Using these relations and the amalgamation properties (6.21) of $\mathcal{D}(A)$ one computes

$$\begin{aligned}\mathcal{D}(a\otimes\varphi)\mathcal{D}(l\otimes\hat{S}(\lambda)) &= \mathcal{D}(al^{(2)}\otimes\varphi^{(2)}\hat{S}(\lambda))\langle l^{(1)},\hat{S}^{-1}(\varphi^{(3)})\rangle\langle l^{(3)},\varphi^{(1)}\rangle\\ &= \mathcal{D}(al^{(2)}\otimes\hat{\Pi}^L(s^{-1}\rightharpoonup\varphi^{(2)})\hat{S}(\lambda))\langle l^{(1)},\hat{S}^{-1}(\varphi^{(3)})\rangle\langle l^{(3)},\varphi^{(1)}\rangle\\ &= \mathcal{D}(al^{(2)}(\varphi^{(2)}\rightharpoonup\mathbf{1})\otimes\hat{S}(\lambda))\langle l^{(3)},\varphi^{(1)}\rangle\langle s^{-1}S^{-1}(l^{(1)}),\varphi^{(3)}\rangle\\ &= \mathcal{D}(a(\hat{\hat{\Pi}}^R(\varphi^{(2)})\rightharpoonup l^{(2)})\otimes\hat{S}(\lambda))\langle l^{(3)},\varphi^{(1)}\rangle\langle s^{-1}S^{-1}(l^{(1)}),\varphi^{(3)}\rangle\\ &= \mathcal{D}(al^{(2)}\otimes\hat{S}(\lambda))\langle l^{(3)},\hat{\hat{\Pi}}^R(\varphi^{(2)})\varphi^{(1)}\rangle\langle s^{-1}S^{-1}(l^{(1)}),\varphi^{(3)}\rangle\\ &= \mathcal{D}(al^{(2)}\otimes\hat{S}(\lambda))\langle l^{(3)}s^{-1}S^{-1}(l^{(1)}),\varphi\rangle\\ &= \mathcal{D}(al^{(1)}\otimes\hat{S}(\lambda))\langle\Pi^L(l^{(2)}),\varphi\rangle=\mathcal{D}(a(\hat{\Pi}^L(\varphi)\rightharpoonup l)\otimes\hat{S}(\lambda))\\ &= \mathcal{D}(aS(\hat{\Pi}^L(\varphi)\rightharpoonup\mathbf{1})l\otimes\hat{S}(\lambda))=\mathcal{D}(a(\mathbf{1}\leftharpoonup\hat{S}^{-1}(\hat{\Pi}^L(\varphi)))l\otimes\hat{S}(\lambda))\\ &= \mathcal{D}(\Pi^L(a(\mathbf{1}\leftharpoonup\hat{\hat{\Pi}}^R(\varphi)))l\otimes\hat{S}(\lambda))=\mathcal{D}(\Pi^L(a(\mathbf{1}\leftharpoonup\hat{\hat{\Pi}}^R(\varphi)))\otimes\hat{\mathbf{1}})\mathcal{D}(l\otimes\hat{S}(\lambda))\\ &= \Pi^L_{\mathcal{D}}(\mathcal{D}(a\otimes\varphi))\mathcal{D}(l\otimes\hat{S}(\lambda)),\qquad a\in A,\varphi\in\hat{A},\end{aligned}\tag{6.25}$$

that is $\mathcal{D}(l\otimes\hat{S}(\lambda))$ is a left integral in $\mathcal{D}(A)$. Similarly to (6.25) one obtains that it is also a right integral, only we have to show that $\mathcal{D}(l\otimes\hat{S}(\lambda))$ is a non-degenerate functional on the dual $\hat{\mathcal{D}}(A)$ of $\mathcal{D}(A)$. The WHA $\hat{\mathcal{D}}(A)$ [3] is the tensor product of $\hat{A}$ and A amalgamated by $\hat{A}^R\simeq A^L$ and $\hat{A}^L\simeq A^R$

$$\hat{\mathcal{D}}(A)\ni\hat{\mathcal{D}}(\varphi\otimes x_LaS^{-1}(x_R))=\hat{\mathcal{D}}(\hat{S}^{-1}(\hat{\mathbf{1}}\leftharpoonup x_R)\varphi(x_L\rightharpoonup\hat{\mathbf{1}})\otimes a)\tag{6.26}$$

with $\varphi\in\hat{A},a\in A,x_{L/R}\in A^{L/R}$ together with the WHA structure maps transposed to that of $\mathcal{D}(A)$ with respect to the non-degenerate pairing

$$\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(b\otimes\psi)\rangle:=\langle\varphi\otimes a,P(b\otimes\psi)\rangle=\langle\hat{P}(\varphi\otimes a),b\otimes\psi\rangle,\quad a,b\in A,\ \varphi,\psi\in\hat{A},\tag{6.27}$$

where $P\colon A\otimes\hat{A}\to A\otimes\hat{A}$ and $\hat{P}\colon\hat{A}\otimes A\to\hat{A}\otimes A$ are the k-linear projections given by the help of separating idempotents of A^L and A^R

$$\begin{aligned}P(b\otimes\psi)&:=b\mathbf{1}^{(1)}S(\mathbf{1}^{(1')})\otimes(\mathbf{1}^{(2')}\rightharpoonup\hat{\mathbf{1}})(\hat{\mathbf{1}}\leftharpoonup S(\mathbf{1}^{(2)}))\psi,\\ \hat{P}(\varphi\otimes a)&:=(\mathbf{1}^{(1')}\rightharpoonup\hat{\mathbf{1}})\varphi(\mathbf{1}^{(1)}\rightharpoonup\hat{\mathbf{1}})\otimes\mathbf{1}^{(2)}a\mathbf{1}^{(2')}.\end{aligned}\tag{6.28}$$

Clearly, $P(A\otimes\hat{A})$ and $\mathcal{D}(A)$ ($\hat{P}(\hat{A}\otimes A)$ and $\hat{\mathcal{D}}(A)$) are isomorphic k-linear spaces and $\mathcal{D}(P(b\otimes\psi))=\mathcal{D}(b\otimes\psi)$ ($\hat{\mathcal{D}}(\hat{P}(\varphi\otimes a))=\hat{\mathcal{D}}(\varphi\otimes a)$) also holds due to the amalgamations in $\mathcal{D}(A)$ ($\hat{\mathcal{D}}(A)$). The two-sided integral $\mathcal{D}(l\otimes\hat{S}(\lambda))$ is non-degenerate if the k-linear map $R_{\mathcal{D}(l\otimes\hat{S}(\lambda))}\colon\hat{\mathcal{D}}(A)\to\mathcal{D}(A)$ is injective, hence bijective due to finite dimensonality. Thus we have to show that if $0=\hat{\mathcal{D}}(\varphi\otimes a)\rightharpoonup\mathcal{D}(l\otimes\hat{S}(\lambda))$ then $0=\hat{\mathcal{D}}(\varphi\otimes a)$. Using the mentioned isomorphisms of the k-linear spaces, the form of the coproduct in $\mathcal{D}(A)$ and the identity $(\hat{\mathbf{1}}\leftharpoonup\Pi^R_{\hat{S}(\sigma^{-1})}(S(\mathbf{1}^{(1)})))\hat{\Pi}^L_{s^{-1}}(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})=\hat{\mathbf{1}}$ we prove later on one computes

$$
\begin{aligned}
&P(l^{(1)}\otimes\hat{S}(\lambda)^{(2)})\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}\otimes\hat{S}(\lambda)^{(1)})\rangle\\
&:=l^{(1)}\mathbf{1}^{(1)}S(\mathbf{1}^{(1')})\otimes(\mathbf{1}^{(2')}\rightharpoonup\hat{\mathbf{1}})(\hat{\mathbf{1}}\leftharpoonup S(\mathbf{1}^{(2)}))\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}\otimes\hat{S}(\lambda)^{(1)})\rangle\\
&=l^{(1)}S(\mathbf{1}^{(1')})\otimes(\mathbf{1}^{(2')}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}S(\mathbf{1}^{(1)})\otimes\hat{S}(\hat{\mathbf{1}}\leftharpoonup S(\mathbf{1}^{(2)}))\hat{S}(\lambda)^{(1)})\rangle\\
&=l^{(1)}S(\mathbf{1}^{(1')})\otimes(\mathbf{1}^{(2')}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}S(\mathbf{1}^{(1)})\mathbf{1}^{(2)}\otimes\hat{S}(\lambda)^{(1)})\rangle\\
&=[lS(\mathbf{1}^{(1)})]^{(1)}\otimes[(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)]^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}([lS(\mathbf{1}^{(1)})]^{(2)}\otimes[(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)]^{(1)})\rangle\\
&=l^{(1)}\otimes\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}\Pi^R_{\hat{S}(\sigma^{-1})}(S(\mathbf{1}^{(1)}))\otimes\hat{\Pi}^L_{s^{-1}}(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)^{(1)})\rangle\\
&=l^{(1)}\otimes\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}\otimes(\hat{\mathbf{1}}\leftharpoonup\Pi^R_{\hat{S}(\sigma^{-1})}(S(\mathbf{1}^{(1)})))\hat{\Pi}^L_{s^{-1}}(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})\hat{S}(\lambda)^{(1)})\rangle\\
&=l^{(1)}\otimes\hat{S}(\lambda)^{(2)}\langle\hat{\mathcal{D}}(\varphi\otimes a),\mathcal{D}(l^{(2)}\otimes\hat{S}(\lambda)^{(1)})\rangle\\
&=l^{(1)}\otimes\hat{S}(\lambda)^{(2)}\langle\hat{P}(\varphi\otimes a),l^{(2)}\otimes\hat{S}(\lambda)^{(1)}\rangle=(R_l\otimes\hat{L}_{\hat{S}(\lambda)})(\hat{P}(\varphi\otimes a)).
\end{aligned}\tag{6.29}
$$

However, the k-linear map $R_l\otimes\hat{L}_{\hat{S}(\lambda)}\colon\hat{A}\otimes A\to A\otimes\hat{A}$ is injective due to the non-degeneracy of the integrals l and λ, hence (6.29) implies that $\hat{P}(\varphi\otimes a)$, or equivalently $\hat{\mathcal{D}}(\varphi\otimes a)$, should be zero if the left hand side of (6.29), or equivalently $\hat{\mathcal{D}}(\varphi\otimes a)\rightharpoonup\mathcal{D}(l\otimes\hat{S}(\lambda))$, is zero.

Finally, the proof of the identity we used in (6.29) is as follows:

$$
\begin{aligned}
&(\hat{\mathbf{1}}\leftharpoonup\Pi^R_{\hat{S}(\sigma^{-1})}(S(\mathbf{1}^{(1)})))\hat{\Pi}^L_{s^{-1}}(\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})=(\hat{\mathbf{1}}\leftharpoonup\Pi^R(S(\mathbf{1}^{(1)})\leftharpoonup\hat{S}(\sigma^{-1})))\hat{\Pi}^L(s^{-1}\mathbf{1}^{(2)}\rightharpoonup\hat{\mathbf{1}})\\
&=(\hat{\mathbf{1}}\leftharpoonup S(S(\mathbf{1}^{(1)})\leftharpoonup\hat{S}(\sigma^{-1})))\hat{S}(\hat{\mathbf{1}}^{(1)})\langle s^{-1}\mathbf{1}^{(2)},\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\hat{\mathbf{1}}^{(1)})\langle s^{-1}\mathbf{1}^{(2)},\hat{\mathbf{1}}^{(2)}(\hat{\mathbf{1}}\leftharpoonup(\sigma^{-1}\rightharpoonup S^2(\mathbf{1}^{(1)})))\rangle\\
&=\hat{S}(\hat{\mathbf{1}}^{(1)})\langle s^{-1}\mathbf{1}^{(2)},\hat{\mathbf{1}}^{(2)}\leftharpoonup(\sigma^{-1}\rightharpoonup S^2(\mathbf{1}^{(1)}))\rangle\\
&=\hat{S}(\hat{\mathbf{1}}^{(1)})\langle(\sigma^{-1}\rightharpoonup S^2(\mathbf{1}^{(1)}))s^{-1}\mathbf{1}^{(2)},\hat{\mathbf{1}}^{(2)}\rangle=\hat{S}(\hat{\mathbf{1}}^{(1)})\langle S^2(\mathbf{1}^{(1)})(\sigma\rightharpoonup s^{-1}\mathbf{1}^{(2)}),\hat{\mathbf{1}}^{(2)}\sigma^{-1}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle S^2(\mathbf{1}^{(1)})(\sigma\rightharpoonup s^{-1}\mathbf{1}^{(2)}),\hat{S}^{-1}(\sigma)\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle S^2(\mathbf{1}^{(1)})(\sigma\rightharpoonup s^{-1}\mathbf{1}^{(2)})\leftharpoonup\hat{S}^{-1}(\sigma),\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle S^2(\mathbf{1}^{(1)})(\sigma\rightharpoonup s^{-1}\mathbf{1}^{(2)}ss^{-1}\leftharpoonup\hat{S}^{-1}(\sigma)),\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle S^2(\mathbf{1}^{(1)})S^4(\mathbf{1}^{(2)})S^4(s^{-1}),\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle\Pi^L(S^2(\mathbf{1}^{(1)})S^4(\mathbf{1}^{(2)})S^4(s^{-1})),\hat{\mathbf{1}}^{(2)}\rangle\\
&=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle S^4(\mathbf{1}^{(2)})S^3(\mathbf{1}^{(1)}),\hat{\mathbf{1}}^{(2)}\rangle=\hat{S}(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma)\langle\mathbf{1},\hat{\mathbf{1}}^{(2)}\rangle=\hat{\mathbf{1}},
\end{aligned}
$$

where we used the properties (5.1–2) of left/right grouplike elements, that a left/right grouplike element leads to an algebra automorphism by the left/right Sweedler action and the Radford formula (6.13). ∎

Appendix A

Here we give examples of finite dimensional WHAs of $A = A^L \otimes A^R \equiv B \otimes B^{op}$ type, where B is a separable k-algebra equipped with a non-degenerate functional $E\colon B \to k$ of index $\mathbf{1}$ (see Appendix of [2]), having antipode of infinite order.

Let $B = M_n(\mathbf{R})$, i.e. a full matrix algebra over the real field, and let $\mathrm{tr}\colon B \to \mathbf{R}$ denote the trace functional with $\mathrm{tr}(\mathbf{1}) = n$. Any invertible element $t \in B$ with $\mathrm{tr}(t^{-1}) = 1$ defines a non-degenerate functional $E\colon B \to \mathbf{R}$ by

$$E(x) := \mathrm{tr}(tx), \qquad x \in B, \tag{A.1}$$

which has index $\mathbf{1}$. Indeed, if $\{e_{ab}\}$ is a set of matrix units then $\{e_i\}_i \equiv \{t^{-1}e_{ab}\}_{(a,b)}$ and $\{f_i\}_i \equiv \{e_{ba}\}_{(a,b)}$ are dual $\mathbf{R}$-bases of B with respect to E, $E(e_i f_j) = \delta_{ij}$, and the index of E is

$$\mathrm{Ind}\, E := \sum_i f_i e_i = \sum_{a,b} e_{ba} t^{-1} e_{ab} = \mathrm{tr}(t^{-1}) \sum_b e_{bb} = \mathbf{1}. \tag{A.2}$$

The Nakayama automorphism θ of E defined by $E(xy) =: E(y\theta(x)); x, y \in B$ is inner,

$$\theta(x) = txt^{-1}, \qquad x \in B \tag{A.3}$$

due to the form (A.1) of E. We can construct the WHA $B \otimes B^{op}$ [2]: it is the $\mathbf{R}$-linear space $B \otimes B$ with structure maps

$$\begin{aligned} (x_1 \otimes x_2)(y_1 \otimes y_2) &:= x_1 y_1 \otimes y_2 x_2, \\ \Delta(x_1 \otimes x_2) &:= \sum_i (x_1 \otimes f_i) \otimes (e_i \otimes x_2), \\ \varepsilon(x_1 \otimes x_2) &:= E(x_1 x_2), \\ S(x_1 \otimes x_2) &:= x_2 \otimes \theta(x_1). \end{aligned} \tag{A.4}$$

Clearly, $S^2 = \theta \otimes \theta$, therefore the form (A.3) of the Nakayama automorphism θ shows that the order of the antipode S is finite iff $t^m \in \mathrm{Center}\, B$ for a certain positive integer m. However, this is not the case for a generic invertible $t \in B = M_n(\mathbf{R})$ with $\mathrm{tr}(t^{-1})$.

Although the order of the antipode is not finite in the generic case, already S^2 is an inner automorphism by a grouplike element in the trivial subalgebra A^T, which, in this case, is equal to A itself. Indeed

$$S^2(x \otimes y) = \theta(x) \otimes \theta(y) = (t \otimes t^{-1})(x \otimes y)(t^{-1} \otimes t), \tag{A.5}$$

and $t \otimes t^{-1} = (t \otimes \mathbf{1}) S(t^{-1} \otimes \mathbf{1})$ with $t \otimes \mathbf{1} = S^2(t \otimes \mathbf{1}) \in A^L$. Therefore $t \otimes t^{-1}$ is a grouplike element in the trivial subalgebra A^T by Prop. 5.4 ii).

Appendix B

Here we give the generalization of the cyclic module [4] $A^{\natural}_{(\sigma,s)}$ for weak Hopf algebras having a modular pair (σ, s) in involution. The details will be published elsewhere.

Let A be a weak Hopf algebra. The pair $(\sigma, s) \in G(\hat{A}) \times G(A)$ of grouplike elements is called a *modular pair* for A if

$$\sigma \rightharpoonup s = s = s \leftharpoonup \sigma, \qquad s \rightharpoonup \sigma = \sigma = \sigma \leftharpoonup s. \tag{B.1}$$

They form a *modular pair in involution* if they implement the square of the antipode

$$S^2(a) = \sigma \rightharpoonup sas^{-1} \leftharpoonup \sigma^{-1}, \quad a \in A. \tag{B.2}$$

Clearly, a modular pair (in involution) is a self-dual notion for WHAs.

The identity (B.2) is a kind of square root of the Radford formula, hence, modular pairs in involution do not exist for arbitrary WHAs. However, there is a wide class of WHAs having such a pair. For example, in a weak Hopf C^*-algebra A there is a canonical grouplike element $g \in A$ implementing S^2 on A [2], hence $(\hat{\mathbf{1}}, g)$ is a modular pair in involution for A. Another example can be given as follows: Let A be a WHA over k and let the WHA $A_G := \langle A^T, G_R(A)\rangle$ be the subWHA of A generated by the trivial subWHA A^T and by (a subgroup of) the right grouplike elements $G_R(A)$ in A. Then $(\hat{\mathbf{1}}, t)$ with $t \in G^T(A)$ defined in (5.34) is a modular pair in involution for A_G, because t implements S^2 for A^T and $G_R(A)$.

Proposition *Let A be a WHA over the field k and $(\sigma, s) \in G(\hat{A}) \times G(A)$ be a modular pair in involution. Let the cochains $C^n_{(\sigma,s)}(A)$, $n \geq 0$ be defined by the n-fold product of the left regular module ${}_AA$, i.e. the k-linear spaces*

$$\begin{aligned} C^0_{(\sigma,s)}(A) &:= A^L \\ C^n_{(\sigma,s)}(A) &:= A \times A \times \ldots \times A \equiv \Delta^{n-1}(\mathbf{1}) \cdot (A \otimes A \otimes \ldots \otimes A). \end{aligned} \tag{B.3}$$

The face operators $\delta^{(n)}_i : C^{n-1}_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$, $0 \leq i \leq n$ are

$$\begin{aligned} \delta^{(1)}_0(x_L) &:= \bar{\Pi}^R(x_L), \\ \delta^{(1)}_1(x_L) &:= x_L s, \\ \delta^{(n)}_0(a_1 \otimes \ldots \otimes a_{n-1}) &:= \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}, \quad 1 < n, \\ \delta^{(n)}_i(a_1 \otimes \ldots \otimes a_{n-1}) &:= a_1 \otimes a_2 \otimes \ldots \otimes \Delta(a_i) \otimes \ldots \otimes a_{n-1}, \quad 1 \leq i < n, 1 < n, \\ \delta^{(n)}_n(a_1 \otimes \ldots \otimes a_{n-1}) &:= a_1 \otimes \ldots \otimes a_{n-2} \otimes \mathbf{1}^{(1)} a_{n-1} \otimes \mathbf{1}^{(2)} s, \quad 1 < n, ; \end{aligned} \tag{B.4}$$

the degeneracy operators $\sigma^{(n)}_i : C^{n+1}_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$, $0 \leq i \leq n$ are

$$\begin{aligned} \sigma^{(0)}_0(a) &:= \Pi^L(a), \\ \sigma^{(n)}_i(a_1 \otimes \ldots \otimes a_{n+1}) &:= a_1 \otimes \ldots \otimes \Pi^L(a_{i+1}) a_{i+2} \otimes \ldots \otimes a_{n+1}, \quad 0 \leq i < n, 0 < n, \\ \sigma^{(n)}_n(a_1 \otimes \ldots \otimes a_{n+1}) &:= a_1 \otimes \ldots \otimes a_{n-1} \otimes \bar{\Pi}^R(a_{n+1}) a_n, \quad 0 < n, \end{aligned} \tag{B.5}$$

and the cyclic operators $\tau_{(n)}: C^n_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$ *are given by*

$$\begin{aligned} \tau_{(0)}(x_L) &:= x_L, \\ \tau_{(n)}(a_1 \otimes \ldots \otimes a_n) &:= \Delta^{(n-1)}(S(a_1 \leftharpoonup \sigma)) \cdot (a_2 \otimes \ldots \otimes a_n \otimes s), \quad n \geq 1. \end{aligned} \tag{B.6}$$

With the definitions (B.3–6) $A^\natural_{(\sigma,s)} \equiv \{C^n_{(\sigma,s)}(A)\}_{n\geq 0}$ *becomes a* Λ*-module, where* Λ *is the cyclic category.*

References

[1] G. Böhm and K. Szlachányi, A coassociative C^*-quantum group with nonintegral dimensions, *Lett. Math. Phys.* **35** (1996), 437

[2] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras I. Integral theory and C^*-structure, *Journal of Alg.* **221** (1999), 385

[3] G. Böhm, Weak C^*-Hopf algebras and their application to spin models, Ph.D. Thesis, Budapest, 1997.

G. Böhm, Doi–Hopf modules over weak Hopf algebras, *Commun. in Alg.* **28** (2000), 4687

[4] A. Connes and H. Moscovici, Cyclic cohomology, Hopf algebras and the modular theory, preprint, math.QA/9905013
A. Connes and H. Moscovici, Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry, preprint, math.DG/0102167

[5] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley & Sons, 1962.

[6] C.W. Curtis and I. Reiner, *Methods of Representation Theory*, John Wiley & Sons, 1990.

[7] P. Deligne and J.S. Milne, Tannakian categories, *Lecture Notes in Math. 900*, p. 101, Springer, 1982.

[8] J. Fuchs, A. Ganchev and P. Vecsernyés, Rational Hopf Algebras: Polynomial equations, gauge fixing, and low-dimensional examples, *Int. J. of Mod. Phys.* **A10** (1995), 3431

[9] L. Kadison, New examples of Frobenius extensions, *AMS Univ. Lecture Series* **14** (1999)

[10] R.G. Larson and M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. of Math.* **91** (1969), 75

[11] G. Mack and V. Schomerus, Quasi Hopf quantum symmetry in quantum theory, *Nucl. Phys.* **B370** (1992), 185

[12] D. Nikshych and L. Vainerman, Finite quantum grupoids and their applications, preprint, math.QA/0001020

[13] D. Nikshych, On the structure of weak Hopf algebras, preprint, math.QA/0106010

[14] R.S. Pierce, *Associative Algebras*, (Graduate Texts in Mathematics 88) Springer-Verlag, 1982.

[15] D.E. Radford, The order of the antipode of a finite dimensional Hopf algebra is finite, *Amer. J. of Math.* **98** (1973), 333

[16] D.E. Radford, Minimal quasitriangular Hopf algebras, *Journal of Alg.* **157** (1993), 285

[17] M.E. Sweedler, *Hopf Algebras*, Benjamin, 1969.

[18] K. Szlachányi, Weak Hopf algebras, in *Operator Algebras and Quantum Field Theory*, eds.: S. Doplicher, R. Longo, J.E. Roberts, and L. Zsidó, International Press, 1996.

[19] P. Vecsernyés, On the quantum symmetry of the chiral Ising model, *Nucl. Phys.* **B415** (1994), 557

[20] Y. Watatani, Index for C^*-subalgebras, *Memoirs of the AMS* **424** (1990)

[21] D.N. Yetter, Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories, *Contemp. Math.* **134** (1992), 325