

MODULE CATEGORIES, WEAK HOPF ALGEBRAS AND MODULAR INVARIANTS

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ABSTRACT. We develop abstract nonsense for module categories over monoidal categories (this is a straightforward categorification of modules over rings). As applications we show that any semisimple monoidal category with finitely many simple objects is equivalent to the category of representations of a weak Hopf algebra (theorem of T. Hayashi) and classify module categories over the fusion category of $\widehat{sl}(2)$ at a positive integer level where we meet once again ADE classification pattern.

1. INTRODUCTION

One considers the notion of an (abelian) monoidal category as a categorification of the notion of a ring. From this point of view it is natural to define a *module category* over a monoidal category as a categorification of the notion of the module over a ring. Such a definition was given by I. Bernstein, and L. Crane and I. B. Frenkel, see [3, 10]. The main point of this paper is to show that module categories is an extremely convenient language. Moreover, this notion is implicitly present in recent developments of such subjects as (1) Boundary Conformal Field Theory, see [2, 14]; (2) Subfactors Theory, see [6, 30]; (3) Theory of weak Hopf algebras, see [28, 34]; (4) Theory of extensions of vertex algebras, see [24]. My own motivation to study this notion comes from the theory of affine Hecke algebras, see [4].

The aim of this paper is to give basic definitions (sections 2 and 3), to present some explanations of the relations with the subjects above (sections 4 and 5) and to give some examples (section 6). Perhaps this paper does not contain new results, but I hope that its point of view, language and some proofs are new.

This paper owes much to many people. I am grateful to Alexander Kirillov, Jr. who explained to me how to do calculations in tensor categories, and for collaboration in paper [24] which strongly influenced this paper; also Lemma 7 below is due to him. I am greatly indebted to Dmitri Nikshych who taught me everything I know about weak Hopf algebras and patiently answered my questions; Lemma 6 below is due to him. My sincere gratitude is due to Pavel Etingof and David Kazhdan for many discussions that significantly clarified my understanding of the subject. Thanks are also due to Leonid Vainerman for bringing reference [19] to my attention.

2. MODULE CATEGORIES

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2.1. Based rings and modules. We will follow the terminology borrowed from [16] and [25]. In what follows all rings, algebras are assumed to be associative with unit. Let \mathbb{Z}_+ be the set of nonnegative integers.

Definition 1. (i) A \mathbb{Z}_+ -basis of an algebra free as a module over \mathbb{Z} is a basis $B = \{b_i\}$ such that $b_i b_j = \sum_k c_{ij}^k b_k$, $c_{ij}^k \in \mathbb{Z}_+$.

(ii) A \mathbb{Z}_+ -ring is an algebra over \mathbb{Z} with unit endowed with a fixed \mathbb{Z}_+ -basis.

(iii) A \mathbb{Z}_+ -module over a \mathbb{Z}_+ -ring A is \mathbb{Z} -free A -module M endowed with a fixed basis $\{m_j\}$ such that $b_i m_j = \sum_k d_{ij}^k m_k$, $d_{ij}^k \in \mathbb{Z}_+$.

(iv) Two \mathbb{Z}_+ -modules M_1, M_2 over A with bases $\{m_i^1\}_{i \in I}$, $\{m_j^2\}_{j \in J}$ are *equivalent* if and only if there exists a bijection $\phi : I \rightarrow J$ such that the induced \mathbb{Z} -linear map $\tilde{\phi}$ of abelian groups M_1, M_2 defined by $\tilde{\phi}(m_i^1) = m_{\phi(j)}^2$ is an isomorphism of A -modules.

(v) The *direct sum* of two \mathbb{Z}_+ -modules M_1, M_2 over A is the module $M_1 \oplus M_2$ over A with the basis being the union of the bases of M_1 and M_2 .

(vi) A \mathbb{Z}_+ -module M over A is *indecomposable* if it is not equivalent to a direct of two nonzero \mathbb{Z}_+ -modules.

(vii) A \mathbb{Z}_+ -submodule of a \mathbb{Z}_+ -module M over A with basis $\{m_i\}_{i \in I}$ is an subset $J \subset I$ such that abelian subgroup of M generated by $\{m_i\}_{i \in J}$ is A -submodule.

(viii) A \mathbb{Z}_+ -module M over A is *irreducible* if any \mathbb{Z}_+ -submodule of M is 0 or M .

Proposition 1. (cf. [18]) For a given \mathbb{Z}_+ -ring A of finite rank over \mathbb{Z} there exist only finitely many irreducible inequivalent \mathbb{Z}_+ -modules over A .

Proof. First of all it is clear that an irreducible \mathbb{Z}_+ -module M over A is of finite rank over \mathbb{Z} . Let $\{m_i\}_{i \in I}$ be the basis of M . Let us consider an element $b := \sum_{b_i \in B} b_i$ of A . Let $b^2 = \sum_i n_i b_i$ and let $N = \max_{b_i \in B} n_i$ (N exists since B is finite). For any $i \in I$ let $b m_i = \sum_{k \in I} d_i^k m_k$ and let $d_i = \sum_{k \in I} d_i^k$. Let $i_0 \in I$ be such that $d = d_{i_0} = \min_{i \in I} d_i$. Let $b^2 m_{i_0} = \sum_{i \in I} c_i m_i$. Calculating $b^2 m_{i_0}$ in two ways — as $(b^2) m_{i_0}$ and as $b(b m_{i_0})$ we have:

$$N d \geq \sum_i c_i \geq d^2$$

and consequently $d \leq N$. So there are only finitely many possibilities for $|I|$, values of c_i and consequently for expansions $b_i m_k$ (since each m_k appears in $b m_{i_0}$). The Proposition is proved. \square

Definition 2. (i) A \mathbb{Z}_+ -ring A with basis $\{b_i\}_{i \in I}$ is called *based ring* if the following conditions hold

(a) There exists subset $I_0 \subset I$ such that $1 = \sum_{i \in I_0} b_i$.

(b) Let $\tau : A \rightarrow \mathbb{Z}$ be the group homomorphism defined by

$$\tau(b_i) = \begin{cases} 1 & \text{if } i \in I_0 \\ 0 & \text{if } i \notin I_0 \end{cases}$$

There exists an involution $i \mapsto \bar{i}$ of I such that induced map $a = \sum_{i \in I} a_i b_i \mapsto \bar{a} = \sum_{i \in I} a_i b_{\bar{i}}$, $a_i \in \mathbb{Z}$ is an anti-involution of ring A and such that

$$\tau(b_i b_j) = \begin{cases} 1 & \text{if } i = \bar{j} \\ 0 & \text{if } i \neq \bar{j}. \end{cases}$$

(ii) A *based module* over a based ring A with basis $\{b_i\}_{i \in I}$ is a \mathbb{Z}_+ -module M with basis $\{m_j\}_{j \in J}$ over A such that $d_{ij}^k = d_{\bar{i}\bar{k}}^j$ where numbers d_{ij}^k are defined in Definition 1 (iii).

(iii) A *unital based ring* is a based ring A such that the set I_0 consists of one element.

Remark 1. (i) It follows easily from definition that $i, j \in I_0$, $i \neq j$ implies that $b_i^2 = b_i$, $b_i b_j = 0$, $\bar{i} = i$.

(ii) It is easy to see that for a given \mathbb{Z}_+ -ring A being based ring is a *property*, not additional structure.

(iii) A different terminology is used in physical literature: our notion of a unital based ring corresponds to the notion of a *fusion rules algebra* (at least in the commutative case) and the notion of a based module corresponds to the notion of a NIM-rep, see [2, 14].

Lemma 1. Let M be a based module over a based ring A . If M is indecomposable as \mathbb{Z}_+ -module over A then M is irreducible as a \mathbb{Z}_+ -module over A .

Proof. Let $\{m_i\}_{i \in I}$ be the basis of M . By the definition of a based module the scalar product on M defined by $(m_i, m_j) = \delta_{ij}$ is invariant with respect to the antiinvolution $a \mapsto \bar{a}$. Hence the orthogonal complement to an A -submodule is again an A -submodule. Finally an orthogonal complement to a \mathbb{Z} -submodule generated by m_k , $k \in K \subset I$ is the \mathbb{Z} -submodule generated by m_j , $j \in I - K$ and therefore the orthogonal complement to a based A -submodule is again a based A -submodule. \square

2.2. Monoidal categories. In this paper we will consider only abelian semisimple categories over a field k with finite dimensional Hom-spaces. If otherwise is not stated explicitly we will assume that the field k is algebraically closed. All functors are assumed to be additive.

Definition 3. (see e.g. [1]) A *monoidal category* consists of the following data: category \mathcal{C} , functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, unit object $\mathbf{1} \in \mathcal{C}$, functorial isomorphisms $r_X : X \otimes \mathbf{1} \rightarrow X$ and $l_X : \mathbf{1} \otimes X \rightarrow X$ subject to the following axioms:

1) Pentagon axiom: the diagram

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \otimes T & \\
 \swarrow a_{X,Y,Z} \otimes id & & \searrow a_{X \otimes Y,Z,T} \\
 (X \otimes (Y \otimes Z)) \otimes T & & (X \otimes Y) \otimes (Z \otimes T) \\
 \downarrow a_{X,Y \otimes Z,T} & & \downarrow a_{X,Y,Z \otimes T} \\
 X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow{id \otimes a_{Y,Z,T}} & X \otimes (Y \otimes (Z \otimes T))
 \end{array}$$

commutes.

2) Triangle axioms: the diagram

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 \searrow r_X \otimes id & & \swarrow id \otimes l_Y \\
 & X \otimes Y &
 \end{array}$$

commutes.

In what follows we will omit from notations associativity and unit isomorphisms what is justified by MacLane coherence theorem, see e.g. [1].

The Grothendieck group $K_0(\mathcal{C})$ of a monoidal category \mathcal{C} is endowed with the structure of \mathbb{Z}_+ -ring: multiplication is induced by \otimes , and \mathbb{Z}_+ -basis consists of classes of simple objects. We will say that a monoidal category \mathcal{C} is a *categorification* of \mathbb{Z}_+ -algebra $K_0(\mathcal{C})$. There exist examples when given \mathbb{Z}_+ -ring admits non unique categorification and when \mathbb{Z}_+ -ring admits no categorifications, see e.g. [35].

Definition 4. (see [1]) (i) Let \mathcal{C} be a monoidal category and X be an object in \mathcal{C} . A *right dual* to X is an object X^* with two morphisms

$$e_X : X^* \otimes X \rightarrow \mathbf{1}, \quad i_X : \mathbf{1} \rightarrow X \otimes X^*$$

such that the compositions

$$\begin{aligned} X &\xrightarrow{i_X \otimes id} X \otimes X^* \otimes X \xrightarrow{id \otimes e_X} X \\ X^* &\xrightarrow{id \otimes i_X} X^* \otimes X \otimes X^* \xrightarrow{e_X \otimes id} X^* \end{aligned}$$

are equal to the identity morphisms.

(ii) A *left dual* to X is an object *X with two morphisms

$$e'_X : X \otimes {}^*X \rightarrow \mathbf{1}, \quad i'_X : \mathbf{1} \rightarrow {}^*X \otimes X$$

such that the compositions

$$\begin{aligned} X &\xrightarrow{i'_X \otimes id} X \otimes {}^*X \otimes X \xrightarrow{id \otimes e'_X} X \\ {}^*X &\xrightarrow{id \otimes i'_X} {}^*X \otimes X \otimes {}^*X \xrightarrow{e'_X \otimes id} {}^*X \end{aligned}$$

are equal to the identity morphisms.

(iii) A monoidal category \mathcal{C} is called *rigid* if every object in \mathcal{C} has right and left duals.

Remark 2. (see [1]) (i) Dual objects are defined canonically, that is if there exists a dual (right or left), it is unique up to a unique isomorphism.

(ii) For any object X of rigid monoidal category \mathcal{C} there are canonical isomorphisms $X = {}^*(X^*) = ({}^*X)^*$.

(iii) Right (and left) duality can be canonically extended to a functor $\mathcal{C} \rightarrow \mathcal{C}^{op}$ where \mathcal{C}^{op} is the opposite category to \mathcal{C} . This functor is equivalence of tensor categories.

Recall that we consider only semisimple categories. One can show that under this assumption for any object X of a rigid monoidal category we have a (noncanonical) isomorphism ${}^*X \simeq X^*$. Hence the Grothendieck group $K_0(\mathcal{C})$ of a rigid monoidal category \mathcal{C} is a based ring.

If we assume that unit object of a rigid monoidal category \mathcal{C} is irreducible then $K_0(\mathcal{C})$ is unital based ring.

Conjecture 1. (Ocneanu rigidity) *For a fixed finite dimensional unital based ring R there are only finitely many rigid monoidal categories \mathcal{C} with $K_0(\mathcal{C}) \simeq R$.*

As far as I know substantial progress in the proof of this Conjecture was achieved by E. Blanchard and A. Wassermann, based on idea of A. Ocneanu. One can also ask if a similar statement is true if we omit rigidity assumption.

Definition 5. Let \mathcal{C} and \mathcal{C}' be two monoidal categories. A *monoidal functor* (F, b, u) is a triple consisting of a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, functorial isomorphism

$b = \{b_{X,Y}\}$, $b_{X,Y} : F(X \otimes Y) = F(X) \otimes F(Y)$, and isomorphism $u : F(\mathbf{1}) = \mathbf{1}$ satisfying the natural compatibilities: the diagrams

$$\begin{array}{ccccc} F((X \otimes Y) \otimes Z) & \xrightarrow{b_{X \otimes Y, Z}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{b_{X, Y} \otimes id} & (F(X) \otimes F(Y)) \otimes F(Z) \\ \downarrow Fa_{X, Y, Z} & & & & \downarrow a_{F(X), F(Y), F(Z)} \\ F(X \otimes (Y \otimes Z)) & \xrightarrow{b_{X, Y \otimes Z}} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{id \otimes b_{Y, Z}} & F(X) \otimes (F(Y) \otimes F(Z)) \end{array}$$

and

$$\begin{array}{ccc} F(\mathbf{1} \otimes X) & \xrightarrow{b_{\mathbf{1}, X}} & F(\mathbf{1}) \otimes F(X) \\ \downarrow F(l_X) & & \downarrow u \otimes id \\ F(X) & \xleftarrow{l_{F(X)}} & \mathbf{1} \otimes F(X) \end{array} \qquad \begin{array}{ccc} F(X \otimes \mathbf{1}) & \xrightarrow{b_{X, \mathbf{1}}} & F(X) \otimes F(\mathbf{1}) \\ \downarrow F(r_X) & & \downarrow id \otimes u \\ F(X) & \xleftarrow{r_{F(X)}} & F(X) \otimes \mathbf{1} \end{array}$$

are commutative.

We give now several well known examples of monoidal categories.

Examples. (i) The category Vec_k of finite dimensional vector spaces over k has a natural structure of a monoidal category where the functor $\otimes = \otimes_k$ is just the usual tensor product. This category is semisimple and rigid. The unit object is a one dimensional space $\mathbf{1}_k$ with fixed basis. The unit object is irreducible. For a monoidal category \mathcal{C} a *fiber functor* is a monoidal functor from \mathcal{C} to Vec_k .

(ii) Let G be an affine group scheme over k . Then category $\text{Rep}(G)$ of finite dimensional rational representations of G has a natural structure of a rigid monoidal category with irreducible unit object, which *is not* semisimple in general. The functor of forgetting the G -action has a natural structure of a fiber functor.

(iii) It is well known that the category of representations of a bialgebra is a monoidal category. More generally, let H be a weak bialgebra, see [34, 28]. Then the category $\text{Rep}(H)$ of H -modules is a monoidal category.

(iv) Let \mathcal{A} be a semisimple abelian category. The category $\text{Fun}(\mathcal{A}, \mathcal{A})$ of functors from \mathcal{A} to \mathcal{A} has a structure of a monoidal category with tensor product induced by composition of functors. This category is semisimple and rigid (duality is given by taking adjoint functor). Its unit object is not irreducible if \mathcal{A} has at least two nonisomorphic irreducible objects.

(v) If characteristic of the base field k is not 2 there are exactly two categories with based ring isomorphic to $K_0(\text{Rep}(\mathbb{Z}/2\mathbb{Z}))$, one is $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$ itself and the second $\text{Rep}(\mathbb{Z}/2\mathbb{Z})^{tw}$ is new. In fact, in such a category there is only one nontrivial associativity constraint (for triple product of nonunit object) and in the category $\text{Rep}(\mathbb{Z}/2\mathbb{Z})^{tw}$ it differs by sign from the one in $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$. Both categories are rigid. The category $\text{Rep}(\mathbb{Z}/2\mathbb{Z})^{tw}$ has no fiber functor.

(vi) More generally, let G be a finite group and consider the category \mathcal{C}_G with (isomorphism classes of) simple objects X_g parametrized by G and the tensor product functor given by $X_{g_1} \otimes X_{g_2} = X_{g_1 g_2}$. The monoidal structures on the category \mathcal{C}_G are parametrized by the group $H^3(G, k^*)$, see e.g. [33, 35].

2.3. Module categories. The following definition is crucial for this paper.

Definition 6. A *module category* over a monoidal category \mathcal{C} is a category \mathcal{M} together with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ and functorial associativity and unit isomorphisms $m_{X, Y, M} : (X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)$, $l_M : \mathbf{1} \otimes M \rightarrow M$ for

any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$ such that the diagrams

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \otimes M & \\
 a_{X,Y,Z} \otimes id \swarrow & & \searrow m_{X \otimes Y, Z, M} \\
 (X \otimes (Y \otimes Z)) \otimes M & & (X \otimes Y) \otimes (Z \otimes M) \\
 \downarrow m_{X, Y \otimes Z, M} & & \downarrow m_{X, Y, Z \otimes M} \\
 X \otimes ((Y \otimes Z) \otimes M) & \xrightarrow{id \otimes m_{Y, Z, M}} & X \otimes (Y \otimes (Z \otimes M))
 \end{array}$$

and

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes M & \xrightarrow{m_{X, \mathbf{1}, Y}} & X \otimes (\mathbf{1} \otimes M) \\
 \downarrow r_X \otimes id & & \downarrow id \otimes l_M \\
 & X \otimes M &
 \end{array}$$

commute.

Remark 3. As far as I know this definition first appeared in Bernstein’s lectures [3] and in the work of L. Crane and I. Frenkel [10]. This notion is implicitly present in Boundary Conformal Field Theory, see e.g. [2, 14, 32]. In this context \mathcal{C} is the fusion category of the corresponding Conformal Field Theory and irreducible objects of \mathcal{M} are “boundary conditions”. It is clear that module categories can be described by certain “6j–symbols” ${}^{(1)}F$ (this description is analogous to the 6j–symbols description of monoidal categories). In Boundary Conformal Field Theory these 6j–symbols appear as coefficients of boundary field operator product expansion. So we consider the notion of module category as a coordinate free version of Boundary Conformal Field Theory. Many examples of module categories (without using this name) were studied in Operator Algebras Theory, see e.g. [6, 30].

The Grothendieck group $K_0(\mathcal{M})$ of a module category \mathcal{M} over a monoidal category \mathcal{C} with basis given by classes of irreducible objects is clearly a \mathbb{Z}_+ –module over the \mathbb{Z}_+ –ring $K_0(\mathcal{C})$.

Lemma 2. Let \mathcal{C} be a rigid monoidal category and \mathcal{M} be a module category over \mathcal{C} . Then for any $X \in \mathcal{C}$, $M_1, M_2 \in \mathcal{M}$ we have canonical isomorphisms

$$\mathrm{Hom}(X \otimes M_1, M_2) \cong \mathrm{Hom}(M_1, {}^*X \otimes M_2), \quad \mathrm{Hom}(M_1, X \otimes M_2) = \mathrm{Hom}(X^* \otimes M_1, M_2).$$

Proof. Clear. \square

The Lemma implies that for a module category \mathcal{M} over a rigid monoidal category \mathcal{C} the Grothendieck group $K_0(\mathcal{M})$ is a based module over the based ring $K_0(\mathcal{C})$.

Examples. (i) Any monoidal category \mathcal{C} has a structure of a module category over itself with associativity and unit isomorphisms induced by the ones from the monoidal structure on \mathcal{C} .

(ii) Let $F : \mathcal{C} \rightarrow \mathrm{Vec}_k$ be a fiber functor. It defines a structure of a module category over \mathcal{C} on the category Vec_k as follows: for $X \in \mathcal{C}$ and $V \in \mathrm{Vec}_k$ we set $X \otimes V := F(X) \otimes_k V$ with associativity and unit isomorphisms defined as compositions:

$$\begin{aligned}
 (X \otimes Y) \otimes V &= F(X \otimes Y) \otimes_k V \xrightarrow{b_{X,Y} \otimes_k id} (F(X) \otimes_k F(Y)) \otimes_k V = \\
 &= F(X) \otimes_k (F(Y) \otimes_k V) = X \otimes (Y \otimes V)
 \end{aligned}$$

and

$$\mathbf{1} \otimes V = F(\mathbf{1}) \otimes_k V \xrightarrow{u \otimes_k id} \mathbf{1}_k \otimes V = V.$$

It is easy to see that conversely, a structure of module category over \mathcal{C} on Vec_k determines a fiber functor on \mathcal{C} (see Section 4 for a more general statement).

(iii) Assume for a moment that the field k is not algebraically closed and consider the category $\mathcal{C} = \text{Vec}_k$. It is easy to see that indecomposable module categories over \mathcal{C} are classified by (finite dimensional) skew fields over k , or equivalently by the Brauer groups of all finite extensions of k .

Definition 7. (i) Let \mathcal{M}_1 and \mathcal{M}_2 be two module categories over a monoidal category \mathcal{C} . A *module functor* from \mathcal{M}_1 to \mathcal{M}_2 is a functor $F : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ together with functorial morphism $c_{X,M} : F(X \otimes M) \rightarrow X \otimes F(M)$ for any $X \in \mathcal{C}$, $M \in \mathcal{M}_1$ such that the diagrams

$$\begin{array}{ccc} & F((X \otimes Y) \otimes M) & \\ & \swarrow^{Fm_{X,Y,M}} \quad \searrow^{c_{X \otimes Y}} & \\ F(X \otimes (Y \otimes M)) & & (X \otimes Y) \otimes F(M) \\ \downarrow^{c_{X,Y \otimes M}} & & \downarrow^{m_{X,Y,F(M)}} \\ X \otimes F(Y \otimes M) & \xrightarrow{id \otimes c_{Y,M}} & X \otimes (Y \otimes F(M)) \end{array}$$

and

$$\begin{array}{ccc} F(\mathbf{1} \otimes M) & \xrightarrow{Fl_M} & F(M) \\ & \searrow^{c_{\mathbf{1},M}} \quad \swarrow^{l_{F(M)}} & \\ & \mathbf{1} \otimes F(M) & \end{array}$$

are commutative.

(ii) We say that a module functor $(F, c_{X,M})$ is *strict* if all morphisms $c_{X,M}$ are isomorphisms.

(iii) Two module categories \mathcal{M}_1 and \mathcal{M}_2 over \mathcal{C} are *equivalent* if there exists a strict module functor from \mathcal{M}_1 to \mathcal{M}_2 which is an equivalence of categories.

(iv) For two module categories \mathcal{M}_1 and \mathcal{M}_2 over a monoidal category \mathcal{C} their *direct sum* is the category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinatewise additive and module structure.

(v) A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories.

Remark 4. For a rigid monoidal category \mathcal{C} any module functor is automatically strict.

Conjecture 2. *For a given rigid monoidal category with finitely many irreducible objects there exists only finitely many inequivalent indecomposable module categories.*

3. MORITA THEORY FOR MODULE CATEGORIES

In this section \mathcal{C} denotes a semisimple monoidal category.

3.1. Algebras in monoidal categories. Definition 8. (cf. e.g. [7]) (i) An *algebra* in a monoidal category \mathcal{C} is an object A of \mathcal{C} endowed with a *multiplication morphism* $m : A \otimes A \rightarrow A$ and a *unit morphism* $e : \mathbf{1} \rightarrow A$ such that the diagrams

$$\begin{array}{ccc}
 & (A \otimes A) \otimes A & \\
 \swarrow^{a_{A,A,A}} & & \searrow^{m \otimes id} \\
 A \otimes (A \otimes A) & & A \otimes A \\
 \downarrow{id \otimes m} & & \downarrow{m} \\
 A \otimes A & \xrightarrow{m} & A
 \end{array}$$

and

$$\begin{array}{ccc}
 & \mathbf{1} \otimes A & \\
 \swarrow{l_A} & & \searrow{e \otimes id} \\
 A & \xrightarrow{m} & A \otimes A \\
 & & \downarrow{m} \\
 & & A
 \end{array}
 \quad
 \begin{array}{ccc}
 & A \otimes \mathbf{1} & \\
 \swarrow{r_A} & & \searrow{id \otimes e} \\
 A & \xrightarrow{m} & A \otimes A \\
 & & \downarrow{m} \\
 & & A
 \end{array}$$

commute.

(ii) A *right module* over an algebra A in a monoidal category \mathcal{C} is an object M of \mathcal{C} together with an *action morphism* $a : M \otimes A \rightarrow M$ such that the diagrams

$$\begin{array}{ccc}
 M \otimes A \otimes A & \xrightarrow{id \otimes m} & M \otimes A \\
 \downarrow{a \otimes id} & & \downarrow{a} \\
 M \otimes A & \xrightarrow{a} & M
 \end{array}
 \quad
 \begin{array}{ccc}
 & M \otimes \mathbf{1} & \\
 \swarrow{r_M} & & \searrow{id \otimes e} \\
 M & \xrightarrow{a} & M \otimes A
 \end{array}$$

commute. A *left module* over A is defined in a similar way.

(iii) A *morphism* between two right modules M_1 and M_2 over A is a morphism $\alpha \in \text{Hom}_{\mathcal{C}}(M_1, M_2)$ such that the diagram

$$\begin{array}{ccc}
 M_1 \otimes A & \xrightarrow{\alpha \otimes id} & M_2 \otimes A \\
 \downarrow{a_1} & & \downarrow{a_2} \\
 M_1 & \xrightarrow{\alpha} & M_2
 \end{array}$$

commutes. It is clear that morphisms between M_1 and M_2 form a k -subspace of $\text{Hom}_{\mathcal{C}}(M_1, M_2)$. We will denote this subspace $\text{Hom}_A(M_1, M_2)$.

Exercise. Prove that if M is a right A -module then $*M$ (but not M^* !) has a natural structure of a left A -module. If M is a left A -module then M^* has natural structure of a right A -module. These two functors are inverse to each other.

We leave to the reader to define morphisms between algebras, ideals, bimodules, tensor products over A etc.

Lemma 3. Let A be an algebra in a monoidal category \mathcal{C} . Right (and similarly left) modules over A with morphisms defined above form an abelian category $\text{Mod}_{\mathcal{C}}(A)$.

Proof. Clear. \square

For any right module M over an algebra A in \mathcal{C} and object X of \mathcal{C} the object $X \otimes M$ has natural structure of A -module with action morphism given by $id \otimes a$. Moreover, for any $X, Y \in \mathcal{C}$ associativity isomorphism between $(X \otimes Y) \otimes M$ and $X \otimes (Y \otimes M)$ is a morphism of A -modules. It is straightforward to check that in

this way we endow $\text{Mod}_{\mathcal{C}}(A)$ with a structure of module category over \mathcal{C} . We will prove in this section that if \mathcal{C} is rigid and has finitely many irreducible objects then any semisimple module category is equivalent to $\text{Mod}_{\mathcal{C}}(A)$ for some algebra A .

Definition 9. (i) An algebra A in \mathcal{C} is *semisimple* if the category $\text{Mod}_{\mathcal{C}}(A)$ is semisimple.

(ii) An algebra A is called *indecomposable* if the module category $\text{Mod}_{\mathcal{C}}(A)$ is indecomposable.

Remark 5. It is easy to see that A is indecomposable in our sense if and only if it is indecomposable in usual sense — it is not isomorphic to a direct sum of nontrivial algebras.

Definition 10. Two algebras A_1 and A_2 in \mathcal{C} are *Morita equivalent* if module categories $\text{Mod}_{\mathcal{C}}(A_1)$ and $\text{Mod}_{\mathcal{C}}(A_2)$ are equivalent.

Remark 6. Another definition (not using the notion of module categories) of Morita equivalence was proposed in [14]. It is easy to see that our definition and the definition of [14] are the same. We hope that our language gives a little bit more flexibility (see e.g. below the proof of Theorem 6).

Let A be an algebra in \mathcal{C} . For any $X \in \mathcal{C}$ the object $X \otimes A$ has a natural structure of right A -module induced by multiplication in A .

Lemma 4. For any A -module L and an object $X \in \mathcal{C}$ we have canonical isomorphism

$$\text{Hom}_A(X \otimes A, L) = \text{Hom}(X, L).$$

Proof. It is clear that unit morphism $e : \mathbf{1} \rightarrow A$ defines canonical isomorphism $\text{Hom}_A(A, L) = \text{Hom}(\mathbf{1}, L)$. Now the isomorphism of Lemma can be obtained as composition

$$\text{Hom}_A(X \otimes A, L) = \text{Hom}_A(A, {}^*X \otimes L) = \text{Hom}(\mathbf{1}, {}^*X \otimes L) = \text{Hom}(X, L).$$

□

Remark 7. In the proof above we used the rigidity of \mathcal{C} . In fact this can be avoided by a more lengthy calculation. We leave this to the reader as an exercise.

We see that modules $X \otimes A$ are projective objects of $\text{Mod}_{\mathcal{C}}(A)$ and any irreducible A -module is a quotient of a module of the form $X \otimes A$. In particular the category $\text{Mod}_{\mathcal{C}}(A)$ has enough projective objects.

3.2. Internal Hom for module categories. In this subsection \mathcal{C} is a semisimple rigid monoidal category and \mathcal{M} is a semisimple module category over \mathcal{C} .

Definition 11. Let M_1 and M_2 be two objects of \mathcal{M} . Their *internal Hom* is an ind-object $\underline{\text{Hom}}(M_1, M_2)$ of \mathcal{C} representing functor $X \mapsto \text{Hom}(X \otimes M_1, M_2)$.

Remark 8. (i) Functor $X \mapsto \text{Hom}(X \otimes M_1, M_2)$ is exact whence existence of $\underline{\text{Hom}}(M_1, M_2)$.

(ii) If both categories \mathcal{C} and \mathcal{M} have finitely many irreducible objects, then $\underline{\text{Hom}}(M_1, M_2)$ is an object of \mathcal{C} .

(iii) By Yoneda's Lemma the object $\underline{\text{Hom}}(M_1, M_2)$ is uniquely defined up to unique isomorphism, so $\underline{\text{Hom}}(?, ?)$ is a bifunctor.

Lemma 5. We have canonical isomorphisms

- (1) $\text{Hom}(X \otimes M_1, M_2) = \text{Hom}(X, \underline{\text{Hom}}(M_1, M_2))$,
- (2) $\text{Hom}(M_1, X \otimes M_2) = \text{Hom}(\mathbf{1}, X \otimes \underline{\text{Hom}}(M_1, M_2))$,
- (3) $\underline{\text{Hom}}(X \otimes M_1, M_2) = \underline{\text{Hom}}(M_1, M_2) \otimes X^*$,
- (4) $\underline{\text{Hom}}(M_1, X \otimes M_2) = X \otimes \underline{\text{Hom}}(M_1, M_2)$.

Proof. Formula (1) is just the definition of $\underline{\text{Hom}}(M_1, M_2)$ and isomorphism (2) is the composition

$$\begin{aligned} \text{Hom}(M_1, X \otimes M_2) &\cong \text{Hom}(X^* \otimes M_1, M_2) = \\ &= \text{Hom}(X^*, \underline{\text{Hom}}(M_1, M_2)) \cong \text{Hom}(\mathbf{1}, X \otimes \underline{\text{Hom}}(M_1, M_2)). \end{aligned}$$

We get isomorphism (3) from the calculation

$$\begin{aligned} \text{Hom}(Y, \underline{\text{Hom}}(X \otimes M_1, M_2)) &= \text{Hom}(Y \otimes (X \otimes M_1), M_2) = \text{Hom}((Y \otimes X) \otimes M_1, M_2) = \\ &= \text{Hom}(Y \otimes X, \underline{\text{Hom}}(M_1, M_2)) = \text{Hom}(Y, \underline{\text{Hom}}(M_1, M_2) \otimes X^*) \end{aligned}$$

and isomorphism (4) from the calculation

$$\begin{aligned} \text{Hom}(Y, \underline{\text{Hom}}(M_1, X \otimes M_2)) &= \text{Hom}(Y \otimes M_1, X \otimes M_2) = \\ &= \text{Hom}(X^* \otimes (Y \otimes M_1), M_2) = \text{Hom}((X^* \otimes Y) \otimes M_1, M_2) = \\ &= \text{Hom}(X^* \otimes Y, \underline{\text{Hom}}(M_1, M_2)) = \text{Hom}(Y, X \otimes \underline{\text{Hom}}(M_1, M_2)). \end{aligned}$$

□

For two objects M_1, M_2 of \mathcal{M} we have the canonical morphism

$$\text{ev}_{M_1, M_2} : \underline{\text{Hom}}(M_1, M_2) \otimes M_1 \rightarrow M_2$$

obtained as the image of id under the isomorphism

$$\text{Hom}(\underline{\text{Hom}}(M_1, M_2), \underline{\text{Hom}}(M_1, M_2)) = \text{Hom}(\underline{\text{Hom}}(M_1, M_2) \otimes M_1, M_2).$$

Let M_1, M_2, M_3 be three objects of \mathcal{M} . Then there is a canonical composition morphism

$$\begin{aligned} (\underline{\text{Hom}}(M_2, M_3) \otimes \underline{\text{Hom}}(M_1, M_2)) \otimes M_1 &= \underline{\text{Hom}}(M_2, M_3) \otimes (\underline{\text{Hom}}(M_1, M_2) \otimes M_1) \xrightarrow{id \otimes \text{ev}_{M_1, M_2}} \\ &\xrightarrow{id \otimes \text{ev}_{M_1, M_2}} \underline{\text{Hom}}(M_2, M_3) \otimes M_2 \xrightarrow{\text{ev}_{M_2, M_3}} M_3 \end{aligned}$$

which produces the *multiplication morphism*

$$\underline{\text{Hom}}(M_2, M_3) \otimes \underline{\text{Hom}}(M_1, M_2) \rightarrow \underline{\text{Hom}}(M_1, M_3)$$

(note that order of factors is opposite to intuitive one!). It is straightforward to check that this multiplication is associative and compatible with the isomorphisms of Lemma 5.

3.3. Main Theorem. In this subsection we assume that \mathcal{C} is a semisimple rigid monoidal category with finitely many irreducible objects and irreducible unit object.

Theorem 1. Let \mathcal{M} be a semisimple indecomposable module category over \mathcal{C} . Then there exists a semisimple indecomposable algebra $A \in \mathcal{C}$ such that the module categories \mathcal{M} and $\text{Mod}_{\mathcal{C}}(A)$ are equivalent.

Proof. Fix any nonzero object M of \mathcal{M} . The multiplication morphism defines a structure of an algebra on $A = \underline{\text{Hom}}(M, M)$. Consider a functor F from \mathcal{M} to \mathcal{C} defined by $N \mapsto \underline{\text{Hom}}(M, N)$. Again the multiplication morphism defines a structure of a right A -module on $\underline{\text{Hom}}(M, N)$ and hence F is a functor from \mathcal{M} to $\text{Mod}_{\mathcal{C}}(A)$. Isomorphism (4) of Lemma 5 defines a structure of a module functor on F (axioms of a module functor follow from compatibility of the multiplication with isomorphism (4) of Lemma 5).

Now we claim that the functor $F : \mathcal{M} \rightarrow \text{Mod}_{\mathcal{C}}(A)$ is an equivalence of categories. We will proceed in steps:

- (1) If $N \neq 0$ then $F(N) \neq 0$.

Indeed, otherwise category \mathcal{M} is decomposable (objects $N \in \mathcal{M}$ such that $F(N) = 0$ clearly form a module subcategory which is a module direct summand of \mathcal{M} thanks to the rigidity of \mathcal{C} , see Lemma 1).

(2) The functor F is injective on Hom's.

This follows immediately from the semisimplicity \mathcal{M} and (1).

(3) The map $F : \text{Hom}(N_1, N_2) \rightarrow \text{Hom}_A(F(N_1), F(N_2))$ is surjective (and hence isomorphism by (2)) for any $N_2 \in \mathcal{M}$ and N_1 of the form $X \otimes M$, $X \in \mathcal{C}$.

Indeed, $F(N_1) = \underline{\text{Hom}}(M, X \otimes M) = X \otimes A$ and the statement follows from the calculation:

$$\begin{aligned} \text{Hom}_A(F(N_1), F(N_2)) &= \text{Hom}_A(X \otimes A, F(N_2)) = \text{Hom}(X, F(N_2)) = \\ &= \text{Hom}(X, \underline{\text{Hom}}(M, N_2)) = \text{Hom}(X \otimes M, N_2) = \text{Hom}(N_1, N_2). \end{aligned}$$

(4) The map $F : \text{Hom}(N_1, N_2) \rightarrow \text{Hom}_A(F(N_1), F(N_2))$ is an isomorphism for any $N_1, N_2 \in \mathcal{M}$.

It is clear that there exist objects $X, Y \in \mathcal{C}$ and an exact sequence

$$Y \otimes M \rightarrow X \otimes M \rightarrow N_1 \rightarrow 0.$$

Hence (4) is consequence of (3).

(5) The functor F is surjective on isomorphism classes of objects of $\text{Mod}_{\mathcal{C}}(A)$.

We know from Lemma 4 that for any object $L \in \text{Mod}_{\mathcal{C}}(A)$ there exists an exact sequence

$$Y \otimes A \xrightarrow{\tilde{f}} X \otimes A \rightarrow L \rightarrow 0$$

for some $X, Y \in \mathcal{C}$. Let $f \in \text{Hom}(Y \otimes M, X \otimes M)$ be the preimage of f under the isomorphism

$$\text{Hom}(Y \otimes M, X \otimes M) = \text{Hom}_A(F(Y \otimes M), F(X \otimes M)) = \text{Hom}_A(Y \otimes A, X \otimes A)$$

and let $N \in \mathcal{M}$ be the cokernel of f . It is clear that $F(N) = L$.

We proved that F is equivalence of categories and proved the Theorem. \square

Remark 9. (i) If a category \mathcal{C} has infinitely many simple objects, one can prove a similar Theorem by considering ind-algebras and ind-modules.

(ii) The proof of the fact that F is an equivalence of categories follows the standard pattern from homological algebra.

Examples. (i) If $\mathcal{M} = \mathcal{C}$ is the ‘‘regular’’ module category and $X \in \mathcal{C}$ it is easy to see that $\underline{\text{Hom}}(X, X) = X \otimes X^*$.

(ii) Let $\mathcal{M} = \text{Vec}_k$ and thus is associated with fiber functor $F : \mathcal{C} \rightarrow \text{Vec}_k$. By the usual Tannakian formalism this induces an equivalence $\mathcal{C} = \text{Rep}(H)$ for some Hopf algebra H . In this case $\underline{\text{Hom}}(k, k) = H^*$ — the dual Hopf algebra together with the natural H -action.

Corollary (of proof). Any semisimple indecomposable algebra in \mathcal{C} is Morita equivalent to the algebra A with $\text{Hom}(\mathbf{1}, A) = k$.

Proof. Indeed it is enough to take for M a simple object of \mathcal{M} . \square

3.4. Example: module categories over $\text{Rep}(G)$. Let G be a finite group. Assume that the base field k is algebraically closed and the characteristic of k does not divide order of G , so the category $\mathcal{C} = \text{Rep}(G)$ is semisimple. In this subsection we classify all module categories over $\text{Rep}(G)$.

Example. Let $H \subset G$ be a subgroup, let $1 \rightarrow k^* \rightarrow \tilde{H} \rightarrow H \rightarrow 1$ be a central extension of H whose kernel is identified with the multiplicative group k^* . Then the category $\text{Rep}^1(\tilde{H})$ of representations V of \tilde{H} such that k^* acts on V via

identity character is a module category over $\text{Rep}(G)$ via usual tensor product. The extensions \tilde{H} as above are in one to one correspondence with elements of the group $H^2(H, k^*)$. For an element $\omega \in H^2(H, k^*)$ we will write $\text{Rep}^1(H, \omega)$ instead of $\text{Rep}^1(\tilde{H})$ where \tilde{H} corresponds to ω .

Theorem 2. (cf. [4]) Any indecomposable module category over $\text{Rep}(G)$ is equivalent to $\text{Rep}^1(H, \omega)$ for some $H \subset G$ and $\omega \in H^2(H, k^*)$. Two module categories $\text{Rep}(H_1, \omega_1)$ and $\text{Rep}(H_2, \omega_2)$ are equivalent if and only if pairs (H_1, ω_1) and (H_2, ω_2) are conjugate under the adjoint action of G .

Proof. First it is easy to see that for any $V \in \text{Rep}^1(\tilde{H})$ we have $\underline{\text{Hom}}(V, V) = \text{Ind}_H^G \text{End}(V)$.

Now let \mathcal{M} be a semisimple indecomposable module category over $\text{Rep}(G)$. By Theorem 1 \mathcal{M} is equivalent to $\text{Mod}_{\text{Rep}(G)}(A)$ for an indecomposable semisimple G -algebra A . It is easy to see that semisimplicity of a G -algebra A implies semisimplicity of A as an algebra in Vec_k , see [4]. So A is just a direct sum of matrix algebras. The group G acts on the set of minimal central idempotents of A and this action is transitive since A is indecomposable. Let e be a minimal central idempotent and let H be its stabilizer in G . It is clear that the subalgebra eAe is H -invariant and $A = \text{Ind}_H^G(eAe)$. The algebra eAe is a matrix algebra and hence $eAe = \text{End}(V)$ for some projective representation of H since all automorphisms of a matrix algebra are inner. The Theorem is proved. \square

Examples. (i) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The only subgroup of G having central extensions is G itself. The group G has exactly one irreducible projective representation (of dimension 2) even though it has two different central extensions. We see that the category $\text{Rep}(G)$ has two different module categories with one irreducible object, hence the category $\text{Rep}(G)$ has an additional fiber functor. In general we see that fiber functors $\text{Rep}(G) \rightarrow \text{Vec}$ are classified by conjugacy classes of pairs (H, ω) such that the category $\text{Rep}^1(H, \omega)$ has only one irreducible object, and in particular the order of H is a square. This result is due to M. V. Movshev, see [27].

(ii) **Definition 12.** ([15]) Two finite groups G_1, G_2 are called *isocategorical* if $\text{Rep}(G_1)$ is equivalent to $\text{Rep}(G_2)$ as a monoidal category.

Of course if the groups G_1, G_2 are isocategorical then the Grothendieck rings $K_0(\text{Rep}(G_1))$ and $K_0(\text{Rep}(G_2))$ are isomorphic as based rings (or equivalently the character tables of G_1 and G_2 are the same). But the property of being isocategorical is much stronger. For example, it is known (see [35], [15]) that the two nonabelian groups of order 8 are not isocategorical (here is a simple proof of this fact: let us calculate the number of fiber functors for these categories; we always have the tautological fiber functor, and it is easy to see from the above that the additional fiber functors are classified by conjugacy classes of subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For the quaternion group we have no such subgroups and for the dihedral group we have two conjugacy classes of such subgroups. So the corresponding monoidal categories have a different number of fiber functors and are not equivalent).

Let G be a finite group. Let $\mathcal{M}(G)$ be a finite set consisting of pairs (H, ω) where $H \subset G$ is a subgroup and $\omega \in H^2(H, k^*)$ is a cohomology class. Let $m : \mathcal{M}(G) \rightarrow \mathbb{N}$ be a function given by $m((H, \omega)) :=$ number of irreducible objects in $\text{Rep}^1(H, \omega)$. It follows from the classification of module categories over $\text{Rep}(G)$ above that for two isocategorical groups G_1 and G_2 there is a bijection $\mathcal{M}(G_1) \rightarrow \mathcal{M}(G_2)$ preserving the function m . In [15] P. Etingof and S. Gelaki described all pairs of

nonisomorphic isocategorical groups and in particular constructed explicit examples. It would be interesting to describe the bijection above for these examples.

3.5. Semisimplicity. In this section we discuss the somewhat subtle question of semisimplicity of algebra A .

Proposition 2. (i) A splitting of the multiplication morphism $m : A \otimes A \rightarrow A$ as a morphism of bimodules over A is sufficient for semisimplicity of algebra A .

(ii) Assume that $\text{Hom}(\mathbf{1}, A) = k$ and let $\varepsilon \in \text{Hom}(A, \mathbf{1})$ be a nonzero morphism. Semisimplicity of A implies that pairing

$$A \otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} \mathbf{1}$$

is nondegenerate; that is, this map defines an isomorphism $A \rightarrow A^*$.

Proof. (i) Indeed, this condition implies that any A -module $L = L \otimes_A A$ is a direct summand of $L \otimes A = L \otimes_A (A \otimes A)$ and consequently is projective.

(ii) It is clear that the map $A \rightarrow A^*$ is a morphism of right A -modules. In the semisimple case the (right or left) A -module A is irreducible since $\text{Hom}_A(A, A) = \text{Hom}(\mathbf{1}, A) = k$ and the map $A \rightarrow A^*$ is isomorphism by the Schur Lemma. \square

Example. Condition (i) is not necessary as the following example shows. Let k be of characteristic 2 and let H be a Hopf algebra dual to the group algebra of $G = \mathbb{Z}/2\mathbb{Z}$. It is easy to see that $k[G]$ is semisimple as an H -algebra but does not satisfy condition (i).

It is likely that the condition of nondegeneracy from (ii) is not sufficient for semisimplicity in general, but I don't know counterexample.

Suppose now that in a category \mathcal{C} we have functorial isomorphisms $\delta_X : X \rightarrow X^{**}$ satisfying

$$\delta_{X \otimes Y} = \delta_X \otimes \delta_Y, \delta_{\mathbf{1}} = id, \delta_{X^*} = (\delta_X^*)^{-1}$$

where for $f \in \text{Hom}(X, Y)$, $f^* \in \text{Hom}(Y^*, X^*)$ is transposed morphism. In this case we can identify X^* and *X , and quantum dimension is defined. The following Theorem is proved in [24] Theorem 4.3 (we work there in a braided category \mathcal{C} with a commutative algebra A , but the proof of 4.3 does not use neither braiding nor commutativity of A):

Theorem 3. Assume that \mathcal{C} is as above. Let A be an algebra in \mathcal{C} with $\text{Hom}(\mathbf{1}, A) = k$ and with a nongenerate pairing defined above. Assume in addition that $\dim(A) \neq 0$. Then condition (i) of Proposition 2 holds and hence the algebra A is semisimple. \square

Remark 10. (i) For monoidal categories coming from Conformal Field Theory condition $\dim(A) \neq 0$ is automatically satisfied since all dimensions are positive.

(ii) Being motivated by the Boundary Conformal Field Theory J. Fuchs and C. Schweigert introduced the notion of a *Frobenius algebra*, see [14]. By definition a Frobenius algebra A is an algebra together with a splitting morphism $\Delta : A \rightarrow A \otimes A$ which is assumed in addition to be coassociative. Note that the splitting constructed in Theorem 3 is automatically coassociative since it is just the dualization of the associative map $m : A \otimes A \rightarrow A$.

(iii) We sketch here a possible way to approach Conjecture 2 for monoidal categories coming from CFT. In view of Lemma 1 and Proposition 1 it would be enough to show that module categories have no deformations. In view of Theorem 1 this reduces to showing that semisimple algebras have no deformations. Such infinitesimal deformations should be described by cohomology of Hochschild complex of

$A - A$ bimodule A . But by Proposition 2 and Theorem 3 above this bimodule is projective, hence its cohomology vanishes.

4. WEAK HOPF ALGEBRAS

4.1. Let H be a bialgebra. The category $\text{Rep}(H)$ has a natural monoidal structure, and many examples of monoidal categories arise in this way. But it is well known that not any monoidal category is equivalent to $\text{Rep}(H)$ for some bialgebra H . The reason is the following: the forgetful functor $\text{Rep}(H) \rightarrow \text{Vec}_k$ is clearly a fiber functor but for a general monoidal category there is no reason to have a fiber functor. Next thing to try is the following: let R be a separable algebra (that is R is a finite direct sum of matrix algebras) and consider the category of finite dimensional R -bimodules $\text{Bimod}(R)$. The category $\text{Bimod}(R)$ is a monoidal category with monoidal structure induced by the tensor product over R .

Definition 13. An R -fiber functor for a monoidal category \mathcal{C} is a monoidal functor $\mathcal{C} \rightarrow \text{Bimod}(R)$.

Let \mathcal{C} be a monoidal category and let $F : \mathcal{C} \rightarrow \text{Bimod}(R)$ be a R -fiber functor. It is natural to expect that Tannakian formalism works in such situation and the functor F induces an equivalence $\mathcal{C} \rightarrow \text{Rep}(H)$, where H is some generalization of a bialgebra. This is indeed true and the corresponding structure on H is a structure of a *weak bialgebra*, see [34]. Recall here the definition of a weak bialgebra.

Definition 14. (see [28, 29, 34]) A weak bialgebra is a finite dimensional vector space H with the structures of an associative algebra $(H, m, 1)$ with multiplication $m : H \otimes H \rightarrow H$ and unit $1 \in H$ and a coassociative coalgebra (H, Δ, ε) with comultiplication $\Delta : H \rightarrow H \otimes H$ and counit $\varepsilon : H \rightarrow k$ such that:

- (i) The comultiplication Δ is a homomorphism of algebras such that

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),$$

- (ii) The counit satisfies the identity:

$$\varepsilon(fgh) = \varepsilon(fg^{(1)})\varepsilon(g^{(2)}h) = \varepsilon(fg^{(2)})\varepsilon(g^{(1)}h)$$

for all $f, g, h \in H$.

The distinction between the definitions of a bialgebra and of a weak bialgebra is the following: in the definition of weak bialgebra it is not assumed that the coproduct preserves the unit and dually it is not assumed that the counit is an algebra homomorphism. The algebra R (denoted by H_t in [28]) is called the *base* of a weak bialgebra H . We refer the reader to [28] and [34] for a review of the theory of weak Hopf algebras (which are weak bialgebras with an antipode). A relation between the theory of weak Hopf algebras and module categories is given by the following

Proposition 3. Let \mathcal{C} be a monoidal category and let R be a separable algebra. There is natural bijection between sets $\{ R\text{-fiber functors} \}$ and $\{ \text{structures of a module category over } \mathcal{C} \text{ on } \text{Rep}(R) \}$.

Proof. The category $\text{Rep}(R)$ has an obvious structure of a module category over $\text{Bimod}(R)$ where the bifunctor $\text{Bimod}(R) \otimes \text{Rep}(R) \rightarrow \text{Rep}(R)$ is the tensor product over R . Hence any R -fiber functor induces a structure of a module category on $\text{Rep}(R)$.

On the other hand, let us reformulate the definition of a module category in the following way: For an abelian category \mathcal{M} , let $\text{Fun}(\mathcal{M}, \mathcal{M})$ denote the category of exact functors $\mathcal{M} \rightarrow \mathcal{M}$ with natural transformations as morphisms. The category

$\text{Fun}(\mathcal{M}, \mathcal{M})$ has a monoidal structure with tensor product being the composition of functors. Now suppose that \mathcal{M} is a module category over a monoidal category \mathcal{C} . Any object $X \in \mathcal{C}$ defines a functor $F_X : \mathcal{M} \rightarrow \mathcal{M}$, $F_X(M) = X \otimes M$. So we have a functor $F : \mathcal{C} \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})$. The associativity isomorphism defines a natural transformation of functors $F_{X \otimes Y} \rightarrow F_X \circ F_Y$, and one checks that the axioms of a module category are equivalent to saying that F is a monoidal functor. Now structures of module category over \mathcal{C} on $\text{Rep}(R)$ are the same as monoidal functors $\mathcal{C} \rightarrow \text{Fun}(\text{Rep}(R), \text{Rep}(R)) = \text{Bimod}(R)$.

We leave it to the reader to check that the two constructions above are mutually inverse. \square

As an easy application we get the following statement:

Theorem 4. Let \mathcal{C} be a semisimple monoidal category with finitely many simple objects. Then there exists an equivalence $\mathcal{C} \cong \text{Rep}(H)$ where H is a weak bialgebra. Moreover, we can assume that the base of H is commutative.

Proof. As we mentioned above, \mathcal{C} is a module category over itself. Choose an algebra R such that \mathcal{C} is equivalent to $\text{Rep}(R)$ as abelian category. By the Proposition above we get R -fiber functor $\mathcal{C} \rightarrow \text{Bimod}(R)$. By the Theorem 1.8 from [34] (see also section 1.3 there) we get an equivalence $\mathcal{C} \rightarrow \text{Rep}(H)$ where H is a weak bialgebra. \square

Remark 11. (i) As it was pointed to me by L. Vainerman, this Theorem was proved previously by T. Hayashi, see [19].

(ii) Some version of the Theorem 4 is known in Operator Algebras theory and in physics, see e.g. [30, 6, 32]. Note that in [30, 6] different terminology is used: there weak Hopf algebras are replaced by closely related objects — double triangular algebras, see [32].

(iii) Note that a weak bialgebra constructed in Theorem 4 is somewhat non-canonical; the canonical object is a bialgebroid over R (see [34]). To get a weak bialgebra from a bialgebroid, one has to choose a separability idempotent in R , see [34]. If R is abelian then a separability idempotent is unique (and our weak bialgebra is canonical); moreover over a field of characteristic 0 the canonical choice of a separability idempotent is possible since a symmetric separability idempotent is unique.

(iv) If the category \mathcal{C} is rigid, one easily defines the antipode map $S : H \rightarrow H$. It is proved in [29] that the map S satisfies the axioms of the antipode in a weak Hopf algebra and so H becomes the weak Hopf algebra.

4.2. Duality for weak Hopf algebras. The definition of a weak bialgebra has a virtue of being selfdual, that is if H is a weak bialgebra then so is H^* , see [28, 34]. In this section we explain the categorical meaning of this duality. Let $\mathcal{M} = \text{Rep}(R)$ be a module category over a monoidal category \mathcal{C} . Let H be the corresponding weak bialgebra constructed by Proposition 3. Consider the category $\mathcal{C}^* := \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of module functors from \mathcal{M} to itself. The category \mathcal{C}^* is an abelian (non-semisimple, in general) monoidal category with composition of functors as a tensor product. Evidently, \mathcal{M} is a module category over \mathcal{C}^* .

Theorem 5. There is a natural monoidal equivalence of categories \mathcal{C}^* and $\text{Rep}(H^*)$.

Proof. Any functor $\text{Rep}(R) \rightarrow \text{Rep}(R)$ is isomorphic to the functor $V \otimes_R ?$ for some R -bimodule V . By definition an object of \mathcal{C}^* is an R -bimodule V together with a functorial isomorphism $c_{X,M} : V \otimes_R (X \otimes_R M) \rightarrow X \otimes_R (V \otimes_R M)$ for

any $X \in \mathcal{C}, M \in \text{Rep}(R)$. Functoriality in the variable M implies that these isomorphisms are induced by R -bimodule isomorphisms $c_X : V \otimes_R X \rightarrow X \otimes_R V$. The isomorphisms c_X should satisfy two conditions: $c_{X_1 \otimes X_2} = (id \otimes c_{X_2})(c_{X_1} \otimes id)$ (the pentagon diagram in Definition 7) and $c_1 = id$ (the triangle axiom in Definition 7). The functoriality in the variable X implies that the isomorphisms c_X are completely defined by the isomorphism c_H (where H is considered as an object of $\mathcal{C} = \text{Rep}(H)$) and moreover by the restriction $c : V \rightarrow H \otimes_R V$ of this isomorphism to $V \otimes 1$ since any vector $x \in X$ is an image of $1 \in H$ under a unique H -module morphism $H \rightarrow X$. One verifies easily that the conditions on the isomorphisms c_X above are equivalent to the condition that c defines a structure of an H -comodule on V (in fact this is only a comodule in the category $\text{Bimod}(R)$, to get a genuine comodule one uses a separability idempotent in R to imbed $H \otimes_R V \subset H \otimes V$). Conversely, let V be an H -comodule (= module over H^*). For any object X of \mathcal{C} define a map $c_X : V \otimes_R X \rightarrow X \otimes_R V$ by the formula

$$c_X(v \otimes x) = \sum_i h_i x \otimes v_i$$

where $v \mapsto \sum_i h_i \otimes v_i$ is a coaction of H on the element $v \in V$. One verifies immediately that this map satisfies the conditions above if it is well defined. So the proof of Theorem 5 is completed modulo the following

Lemma 6. (D. Nikshych) The map c_X above is a well defined map of R -bimodules.

Proof. We will use in the proof the notations from [28].

(1) The map c_X is well defined: let $z \in H_t$ be an arbitrary element. It acts on $v \in V$ by the formula $v \mapsto \sum_i \varepsilon(h_i z) v_i$, see [28] 2.4. We need to prove an equality $c_X(v \otimes zx) = c_X(\sum_i \varepsilon(h_i z) v_i \otimes x)$, or equivalently

$$\sum_i h_i z x \otimes v_i = \sum_i \varepsilon(h_i^{(1)} z) h_i^{(2)} x \otimes v_i$$

which follows from the identity $\varepsilon(h_i^{(1)} z) h_i^{(2)} = h_i z$, see [28] Proposition 2.2.1 (v).

To prove that the map c_X is a morphism of R -bimodules, we need to identify H_s and H_t^{op} . For this we will use $\varepsilon_s|_{H_t}$ and $\varepsilon_t|_{H_s}$ (in the case of weak Hopf algebras this coincides with the usual identification via the antipode).

(2) The map c_X is a map of H_t -modules: this reduces to the equality

$$\sum_i z h_i x \otimes v_i = \sum_i \varepsilon(\varepsilon_s(z) h_i^{(1)}) h_i^{(2)} x \otimes v_i$$

for $z \in H_t, v \in V, x \in X$ which is a consequence of the known identity $zh = \varepsilon(\varepsilon_s(z) h^{(1)}) h^{(2)}$ for $h \in H, z \in H_t$.

(3) The map c_X is a map of H_s -modules: this reduces to the equality

$$\sum_i \varepsilon(h_i^{(1)} \varepsilon_t(z)) h_i^{(2)} x \otimes v_i = \sum_i h_i z x \otimes v_i$$

for $z \in H_s, v \in V, x \in X$ which is a consequence of the known identity $hz = \varepsilon(h^{(1)} \varepsilon_t(z)) h^{(2)}$ for $h \in H, z \in H_s$.

The Lemma and the Theorem are proved. \square

Remark 12. There is an elementary description of the category \mathcal{C}^* in terms of the algebra A : the category \mathcal{C}^* is equivalent to the category of the bimodules over A with \otimes_A as a tensor product (note that this tensor product is exact since the categories of left and right A -modules are semisimple). Furthermore if the category \mathcal{C} is braided then this is the same as category of left modules over $A \otimes$

A^{op} where A^{op} is the opposite algebra of A (we use the braiding to define A^{op} and the multiplication in the tensor product of algebras). This shows that the tensor category of A -bimodules depends only on Morita equivalence class of A and sometimes allows one to describe explicitly simple objects of \mathcal{C}^* .

4.3. Ocneanu cells. It is convenient to say that the category \mathcal{M} is a right module category over the tensor category \mathcal{C}_{op}^* (the category \mathcal{C}_{op}^* is the same as the category \mathcal{C}^* but the tensor product is different: $X \otimes_{op} Y := Y \otimes X$) and consider the definition of \mathcal{C}^* as associativity constraint $(X \otimes M) \otimes F \rightarrow X \otimes (M \otimes F)$ for $X \in \mathcal{C}, M \in \mathcal{M}, F \in \mathcal{C}_{op}^*$. So we have three categories $\mathcal{C}, \mathcal{M}, \mathcal{C}_{op}^*$, four bifunctors $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}, \mathcal{M} \times \mathcal{C}_{op}^* \rightarrow \mathcal{M}, \mathcal{C}_{op}^* \times \mathcal{C}_{op}^* \rightarrow \mathcal{C}_{op}^*$, five associativity constraints and six hexagon axioms. This situation was axiomatized by A. Ocneanu and the corresponding structure constants are known under the name “Ocneanu cells”, see e.g. [32]. So we consider our formalism above as a coordinate free version of Ocneanu cells.

4.4. Dynamical twists in group algebras. Let G be a finite group and $A \subset G$ be an abelian subgroup. Then $\text{Rep}(A)$ is a module category over $\text{Rep}(G)$ and so we have the R -fiber functor F where R is a group algebra of A . In [17] P. Etingof and D. Nikshych studied all possible structures of tensor functor on the functor F (= dynamical twists of the corresponding weak Hopf algebra up to gauge equivalence) and showed that they are classified by the isomorphism classes of “dynamical data”, see [17], Theorem 6.6 for precise statement and examples. From our point of view they studied module categories \mathcal{M} over $\text{Rep}(G)$ such that $K_0(\mathcal{M}) = K_0(\text{Rep}(A))$ as based modules over $K_0(\text{Rep}(G))$. Using Theorem 2 we see that this is equivalent to looking for pairs (H, ω) such that simple objects of $\text{Rep}^1(H, \omega)$ are numbered by $A^* := \text{Hom}(A, \mathbb{C}^*)$ and such that $\dim \text{Hom}_{\bar{H}}(X \otimes V_\chi, V_\psi) = \dim \text{Hom}_A(X \otimes \chi, \psi)$ for any $X \in \text{Rep}(G), \chi, \psi \in A^*$ where V_χ, V_ψ are simple objects in $\text{Rep}^1(H, \omega)$ corresponding to χ and ψ . Using the Frobenius reciprocity we see that this is equivalent to having isomorphisms $\text{Ind}_H^G(V_\psi \otimes V_\chi^*) = \text{Ind}_A^G(\psi \chi^{-1})$ of G -modules for any $\chi, \psi \in A^*$. But this is precisely the definition of dynamical data from [17]. So we see that Theorem 6.6 of [17] is essentially a special case of Theorem 2 above. So we consider Theorem 2 as a generalization of [17].

5. RESULTS OF BÖCKENHAUER, EVANS AND KAWAHIGASHI

Let \mathcal{C} be a rigid monoidal category. In this section we assume in addition that \mathcal{C} is *braided*, see e.g. [1]. In this case Böckenhauer, Evans and Kawahigashi (inspired by A. Ocneanu) proved a number of remarkable results, see [6] and references therein. They worked in the realm of Operator Algebra Theory. We are going to translate their results to a categorical language. We will not give proofs (but note that proofs in [6] are calculations in the weak Hopf algebra attached to a module category over a monoidal category and so should work in the general categorical situation), so the skeptical reader may consider Claims below as Conjectures (but these Conjectures are Theorems in a case when monoidal and module categories under consideration appear in the context of subfactor theory; this class is large enough, for example it includes all fusion categories of $\widehat{sl}(n)$ thanks to the work of A. Wasserman, see [36]). In any case categorical proofs of statements below would be highly desirable.

5.1. α -induction. Let \mathcal{M} be a module category over \mathcal{C} and let \mathcal{C}^* be the corresponding dual category, see 4.2. We will assume that all the categories \mathcal{C} , \mathcal{M} , \mathcal{C}^* are semisimple, the unit object of \mathcal{C} is irreducible, and the category \mathcal{M} is indecomposable (so the unit object of \mathcal{C}^* is also irreducible). Assume that category \mathcal{C} is braided; let $\beta_{V,W} : V \otimes W$ denote braiding and let $\tilde{\beta}_{W,V}$ denote opposite braiding (that is $\beta\tilde{\beta} = Id$). We have two tensor functors $\alpha^+, \alpha^- : \mathcal{C} \rightarrow \mathcal{C}^*$ defined as follows. For any $X \in \mathcal{C}$ we have $\alpha^+(X) = \alpha^-(X) = V \otimes ?$ as functors but module functors structures are different: we set $c_{Y,M}^{\alpha^+(X)} = \beta_{X,Y} \otimes id : \alpha^+(X)(Y \otimes M) = X \otimes Y \otimes M \rightarrow Y \otimes X \otimes M = Y \otimes \alpha^+(X)(M)$ and $c_{Y,M}^{\alpha^-(X)} = \tilde{\beta}_{X,Y} \otimes id$. One checks immediately that this defines a module functors structure using the hexagon axiom. Moreover, α^+ and α^- are tensor functors again thanks to the hexagon axiom (here a tensor structure on α^\pm is given by the associativity constraint: $\alpha^\pm(X \otimes Y) = X \otimes Y \otimes ? = \alpha^\pm(X) \circ \alpha^\pm(Y)$).

Let \mathcal{C}_+^* (resp. \mathcal{C}_-^*) denote the additive subcategory of \mathcal{C}^* whose objects are subquotients (= direct summands) of $\alpha^+(X)$ (resp. $\alpha^-(X)$) for all $X \in \mathcal{C}$. Clearly \mathcal{C}_+^* and \mathcal{C}_-^* are monoidal subcategories of \mathcal{C}^* . One checks easily that the braiding $\beta_{X,Y}$ defines an isomorphism of module functors $\alpha^+(X) \circ \alpha^-(Y) \simeq \alpha^-(Y) \circ \alpha^+(X)$.

Proposition 4. The braiding above restricts to a well defined functorial “relative” braiding $\beta_{F,G}^* : F \circ G \rightarrow G \circ F$ for all $F \in \mathcal{C}_+^*$, $G \in \mathcal{C}_-^*$.

Proof. Let $F \in \mathcal{C}^*$ be a module functor. The module structure on F defines a morphism of functors $F \circ \alpha^\pm(X) \rightarrow \alpha^\pm(X) \circ F$. We will say that F commutes with $\alpha^\pm(X)$ if this morphism is a morphism of module functors. Clearly if F commutes with $\alpha^\pm(X)$ then the same is true for any subquotient of F . One verifies easily that each functor of the form $\alpha^+(X)$ commutes with any functor of the form $\alpha^-(Y)$ and vice versa. So the isomorphism $\alpha^+(X) \circ \alpha^-(Y) \simeq \alpha^-(Y) \circ \alpha^+(X)$ is functorial in the variables $\alpha^+(X)$ and $\alpha^-(Y)$ (that is commutes with any endomorphism of these functors) whence we get the Proposition. \square

In particular consider an additive subcategory $\mathcal{C}_0^* := \mathcal{C}_+^* \cap \mathcal{C}_-^* \subset \mathcal{C}^*$. By the Proposition it has a structure of braided category.

5.2. Modular invariants. Assume in addition that \mathcal{C} is a ribbon category. In this case one defines (see e.g. [1]) the operators S and T acting on the complexified Grothendieck group $K_0(\mathcal{C}) \otimes \mathbb{C}$. Let $Irr(\mathcal{C}) = \{\lambda, \mu, \dots\}$ be the set indexing simple objects in \mathcal{C} . Then S and T can be considered as matrices with rows and columns indexed by λ, μ, \dots by using the basis $[X_\lambda], [X_\mu], \dots$ of $K_0(\mathcal{C})$ consisting of classes of simple objects X_λ, X_μ, \dots . Consider the matrix $Z_{\lambda,\mu} = \dim \text{Hom}_{\mathcal{C}^*}(\alpha^+(X_\lambda), \alpha^-(X_\mu))$.

Claim 1. The matrix $Z_{\lambda,\mu}$ commutes with the matrices S and T .

The matrix $Z_{\lambda,\mu}$ evidently has the properties $Z_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ and $Z_{0,0} = 1$ where $0 \in Irr(\mathcal{C})$ is the index corresponding to the trivial object of \mathcal{C} . Such matrices are called *modular invariants* (under assumption that S is invertible) and were extensively studied, see e.g. [18] and references therein.

5.3. Nondegenerate case. Assume in addition that the category \mathcal{C} is modular, that is the matrix S is nondegenerate. In this case additional results can be established.

Claim 2. The category \mathcal{C}^* is generated by \mathcal{C}_+^* and \mathcal{C}_-^* , that is any object of \mathcal{C}^* is a subquotient of $\alpha^+(X) \circ \alpha^-(Y)$ for some $X, Y \in \mathcal{C}$.

Claim 3. The number of irreducible objects in the category \mathcal{C}^* is $Tr(ZZ^t) = \sum_{\lambda, \mu} Z_{\lambda, \mu}^2$. Moreover the Grothendieck ring $K_0(\mathcal{C}^*)$ is isomorphic to the direct sum of matrix algebras of sizes $Z_{\lambda, \mu}$.

Recall that the characters (= homomorphisms to \mathbb{C}) of $K_0(\mathcal{C})$ are naturally labelled by $\lambda \in Irr(\mathcal{C})$ via the matrix S , see [1]. The abstract $K_0(\mathcal{C}) \otimes \mathbb{C}$ -module $K_0(\mathcal{M}) \otimes \mathbb{C}$ is a direct sum of one dimensional modules, and we refer to the corresponding (multi)subset of $Irr(\mathcal{C})$ as to the (multi)set of *exponents* of \mathcal{M} . On the other hand, the multiset containing λ with multiplicity $Z_{\lambda, \lambda}$ is called the set of exponents of the modular invariant Z .

Claim 4. The number of irreducible objects in the category \mathcal{M} is $Tr(Z) = \sum_{\lambda} Z_{\lambda, \lambda}$. Moreover, the set of exponents of \mathcal{M} coincides with the set of exponents of Z .

5.4. The centers. In this section we translate to our language some results of [5]. Recall that there exists a semisimple algebra $A \in \mathcal{C}$ such that \mathcal{M} is equivalent to $Mod_{\mathcal{C}}(A)$.

Definition 15. The *left center* of A is a maximal subobject $B^+ \subset A$ such that the diagram below commutes:

$$\begin{array}{ccc} B^+ \otimes A & \xrightarrow{\beta_{B^+, A}} & A \otimes B^+ \\ & \searrow m & \swarrow m \\ & & A \end{array}$$

The *right center* of A is a maximal subobject $B^- \subset A$ such that the diagram commutes:

$$\begin{array}{ccc} B^- \otimes A & \xrightarrow{\tilde{\beta}_{B^-, A}} & A \otimes B^- \\ & \searrow m & \swarrow m \\ & & A \end{array}$$

It is clear that both B^+ and B^- are well defined. Note that there is no reason for the equality $B^+ = B^-$ in general (and there are examples when B^+ is not isomorphic to B^- even as an object of \mathcal{C} !). Also it is clear that B^{\pm} are commutative subalgebras of A . It is not difficult to give a definition of B^{\pm} in terms of the category \mathcal{M} : for example, B^+ is the universal object $B \in \mathcal{C}$ endowed with a functorial morphism $z_M : B \otimes M \rightarrow M$ for any $M \in \mathcal{M}$ such that the diagram

$$\begin{array}{ccc} (B \otimes X) \otimes M & \xrightarrow{m_{B, X, M}} & B \otimes (X \otimes M) \\ \beta_{B, X} \otimes id \downarrow & & \downarrow z_{X \otimes M} \\ (X \otimes B) \otimes M & \xrightarrow{(id \otimes z_M) \circ m_{X, B, M}} & X \otimes M \end{array}$$

commutes for any $M \in \mathcal{M}, X \in \mathcal{C}$. This means that the notion of the centers is Morita invariant (that is depends only on the class of Morita equivalence of algebra A).

Let $\mathcal{M}^{\pm} = Mod_{\mathcal{C}}(B^{\pm})$ and let Z^{\pm} denote the corresponding modular invariants. Note that Z^{\pm} are “type I modular invariants”, that is considered as hermitian forms on $K_0(\mathcal{C})$ they are sums of squares of linear combinations of characters, see [24].

Furthermore let $\text{Mod}_{\mathcal{C}}^0(B^\pm)$ denote the tensor category of representations of the “vertex algebra” B^\pm (see [24], this category was denoted $\text{Rep}^0(B^\pm)$ there).

Claim 5. We have tensor equivalences $\text{Mod}_{\mathcal{C}}^0(B^+) = \mathcal{C}_0^* = \text{Mod}_{\mathcal{C}}^0(B^-)$. The modular invariants Z^\pm are “type I parents” of the modular invariant Z . In particular the structure of B^+ as an object of \mathcal{C} is given by the first (vacuum) row of Z and the structure of B^- as an object of \mathcal{C} is given by the first column of Z .

We refer the reader to [5, 11] for the discussion of type I and type II modular invariants and the notion of type I parents (due to G. Moore and N. Seiberg).

5.5. Problem. We would like to close this section by the following problem. Let $\lambda \mapsto \bar{\lambda}$ be the involution of $\text{Irr}(\mathcal{C})$ induced by the duality: $X_\lambda^* \simeq X_{\bar{\lambda}}$. It is well known that the matrix $Z_{\lambda,\mu} = \delta_{\lambda,\bar{\mu}}$ is a modular invariant (“charge conjugation” invariant, see e.g. [11]).

Problem. What is a construction of a module category corresponding to this modular invariant?

Since the charge conjugation modular invariant exists quite generally, one should expect that there exists a very general construction of this kind. It is clear from the discussion above that the number of simple objects in this module category should be equal to the number of selfdual simple objects in \mathcal{C} .

6. MODULE CATEGORIES OVER FUSION CATEGORY OF $\widehat{sl}(2)$

The celebrated result of Capelli-Itzykson-Zuber [8] and Kato [22] states that $\widehat{sl}(2)$ -modular invariants are classified by simply laced Dynkin diagrams. On the other hand we know that modular invariants can be constructed from module categories. The aim of this section is to prove that classification of module categories over fusion categories of $\widehat{sl}(2)$ is exactly the same: indecomposable module categories are classified by simply laced Dynkin diagrams.

6.1. Monoidal category \mathcal{C}_l . Let l be an positive integer. Let \mathcal{C}_l be the category of representations of $\widehat{sl}(2)$ on the level l , see e.g. [21]. This category has a natural structure of a monoidal category (fusion product), see e.g. [13]. Moreover the category \mathcal{C}_l is modular category (= braided, balanced, with invertible S -matrix), see e.g. [1]. The category \mathcal{C}_l is semisimple and has $l + 1$ simple objects denoted by V_0, V_1, \dots, V_l where subscript is highest weight. The object V_0 is the unit object and the structure of $K_0(\mathcal{C}_l)$ is completely determined by rules

$$[V_1][V_i] = [V_i][V_1] = [V_{i-1}] + [V_{i+1}], \quad 1 \leq i < l; \quad [V_1][V_l] = [V_l][V_1] = V_{l-1}.$$

In particular the ring $K_0(\mathcal{C}_l)$ is generated by $[V_1]$. The relations above imply that $[V_l][V_l] = [V_0]$, that is V_l is an invertible object (or “simple current” in physical language).

It is proved by T. Kerler, see [20, 23], that there are exactly two monoidal categories with the Grothendieck ring isomorphic to $K_0(\mathcal{C}_l)$, first is \mathcal{C}_l itself and second is a twisted version \mathcal{C}_l^{tw} of \mathcal{C}_l (the twist comes from the $\mathbb{Z}/2\mathbb{Z}$ grading of $K_0(\mathcal{C}_l)$).

The subcategory of \mathcal{C}_l additively generated by the objects V_0 and V_l is the monoidal subcategory of \mathcal{C}_l with the Grothendieck ring isomorphic to $K_0(\text{Rep}(\mathbb{Z}/2\mathbb{Z}))$. The structure of this category is determined by the following

Lemma 7. (A. Kirillov, Jr.) The monoidal category generated by V_0 and V_l is equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$ for even l and to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})^{tw}$ for odd l .

Proof. The braiding morphism $\beta : V_l \otimes V_l \rightarrow V_l \otimes V_l$ is just a number since $\text{Hom}(V_0, V_0)$ is one dimensional. This number was computed in [24], Lemma 7.6 and it equals to $e^{3\pi il/2}$. So this number equals to ± 1 for even l and $\pm i$ for odd l . Now Example 2.5.3 of [33] implies the Lemma (or better we suggest the reader to deduce the Lemma from the Hexagon axiom as an useful exercise). \square

Remark 13. It is easy to see that the subcategory generated by V_0 and V_l in \mathcal{C}_l^{tw} is always equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$ (since for even l the twist does not touch associativity morphisms in this category and for odd l the twist untwists the category $\text{Rep}(\mathbb{Z}/2\mathbb{Z})^{tw}$ back to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$).

6.2. Based modules over $K_0(\mathcal{C}_l)$. Indecomposable based modules over $K_0(\mathcal{C}_l)$ were classified by P. Di Francesco and J.-B. Zuber, see [12], and independently by P. Etingof and M. Khovanov, see [16]. Such modules are in one to one correspondence with simply laced Dynkin diagrams (possibly with loops) with Coxeter number $h = l + 2$. The correspondence is given as follows: let I be the set of vertices of a Dynkin diagram and consider the matrix $A = (a_{ij})_{i,j \in I}$ given by a_{ij} = number of edges joining vertices i and j (in particular $a_{ii} = 0$ for vertices i without loops). Consider a free \mathbb{Z} -module M with a basis labelled by I . Then letting $[V_1]$ act on M via the matrix A we get a well defined structure of a based module over $K_0(\mathcal{C}_l)$ on M .

Recall (see e.g. [16]) that there are three series of simply laced Dynkin diagrams with loops: A_n, D_n and T_n (“tadpole”, the only diagram with loops) and three exceptionals E_6, E_7, E_8 . The Coxeter numbers of these diagrams are respectively $n + 1, 2n - 2, 2n + 1, 12, 18, 30$. We are going to prove that for each type except T_n there exists a unique module category over \mathcal{C}_l which categorifies the corresponding based module. For type T_n such a module category does not exist.

6.3. Classification of module categories over \mathcal{C}_l . We will say that an indecomposable module category \mathcal{M} over \mathcal{C}_l is of the type A_n, D_n, E_n, T_n if the based module $K_0(\mathcal{M})$ over $K_0(\mathcal{C}_l)$ is of the type A_n, D_n, E_n, T_n via the correspondence in 6.2. In this section we are going to prove the following

Theorem 6. For a simply laced Dynkin diagram X of type A, D, E with Coxeter number h there exists a unique module category of type X over \mathcal{C}_{h-2} . The module category of type T_n over \mathcal{C}_{2n-1} does not exist.

Proof. First we give a construction of all the module categories:

(1) Type A_n . This module category is just the “regular representation” of \mathcal{C}_l that is \mathcal{C}_l itself considered as a module category over itself.

(2) Type D_n . Let l be an even number greater than 2. By Lemma 7 we have that the subcategory of \mathcal{C}_l with irreducible objects V_0, V_l is equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$, in particular the object $A = V_0 \oplus V_l$ has a structure of semisimple algebra. One verifies easily that the corresponding module category $\text{Mod}_{\mathcal{C}_l}(A)$ has Grothendieck group isomorphic to based module of type $D_{l/2+2}$ (see e.g. [24] for the case of $l = 4m$; the case $l = 4m + 2$ is completely analogous).

(3) Types E_6, E_8 . The conformal embedding $\widehat{sl}(2)_{10} \subset \widehat{sp}(4)_1$ determines a structure of a semisimple algebra on the object $A = V_0 \oplus V_6$ of the category \mathcal{C}_{10} , and one verifies easily that the corresponding module category $\text{Mod}_{\mathcal{C}_{10}}(A)$ has Grothendieck group isomorphic to the based module of type E_6 , see [24]. Similarly, the conformal embedding $\widehat{sl}(2)_{28} \subset (\widehat{G}_2)_1$ defines a structure of a semisimple algebra on the object

$A = V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}$ and the corresponding module category has Grothendieck group isomorphic to the based module of type E_8 , see *loc. cit.*

(4) Type E_7 . In this case $l = 16$ and there is no conformal embedding of $\widehat{sl}(2)_{16}$ in any other affine Lie algebra. Instead there exists a conformal embedding $\widehat{sl}(2)_{16} \oplus \widehat{sl}(3)_6 \subset (\widehat{E_8})_1$, see [11], Chapter 17. The category of representations of $\widehat{sl}(2)_{16} \oplus \widehat{sl}(3)_6$ is equivalent to the “tensor product of tensor categories” $\mathcal{C} = \mathcal{C}_{16} \otimes \mathcal{C}(\widehat{sl}(3)_6)$ where $\mathcal{C}(\widehat{sl}(3)_6)$ is a category of integrable $\widehat{sl}(3)$ -modules on level 6 (we leave to the reader the definition of tensor product of monoidal categories and the proof of this statement). The conformal embedding above defines a semisimple algebra A in the category \mathcal{C} ; see [11] 17.109 for the structure of A as an object of \mathcal{C} . Next one calculates that the corresponding module category $\text{Mod}_{\mathcal{C}}(A)$ has 24 irreducible objects. There are two ways to perform this calculation: one using explicit fusion rules for $\widehat{sl}(2)_{16}$ and $\widehat{sl}(3)_6$ similarly to the calculations for D_n, E_6, E_8 , but this is difficult since the category \mathcal{C} has $17 \cdot 28 = 476$ irreducible objects. The second way is to use Claim 4 above (the results of [36] allow one to apply the results of [6] to the category \mathcal{C}) which gives the desired result from one look at [11] 17.109); we prefer this second way but would like to stress that the usage of Claim 4 can be avoided. Now the category \mathcal{C} contains the monoidal subcategory $\mathcal{C}_{16} \otimes \mathbf{1}$ which is equivalent to \mathcal{C}_{16} . This implies that the category \mathcal{C}_{16} has a module category with 24 irreducible objects. Now the results of 6.2 shows that any indecomposable module category over \mathcal{C}_{16} has either 17 (type A_{17}) or 10 (type D_{10}) or 7 (type E_7) irreducible objects. Since there are only two decompositions $24 = 17+7 = 10+7+7$, we see immediately that a module category over \mathcal{C}_{16} of type E_7 does exist. We note that the modular invariants arguments (see [11] 17.6) or explicit calculation show that \mathcal{C}_{16} -module category $\text{Mod}_{\mathcal{C}}(A)$ is a direct sum of the category of type D_{10} and two categories of type E_7 .

Now we prove that to any simply laced Dynkin diagram with loops corresponds at most one module category. Let M be the object of the module category corresponding to the end of the longest leg of the Dynkin diagram. It is easy to calculate the structure of $A = \underline{\text{Hom}}(M, M)$ as an object of \mathcal{C}_l . We get the following table from [24]:

Diagram	$l = h - 2$	A
A_n	$n - 1$	V_0
D_n	$2n - 4$	$V_0 \oplus V_l$
T_n	$2n - 1$	$V_0 \oplus V_l$
E_6	10	$V_0 \oplus V_6$
E_7	16	$V_0 \oplus V_8 \oplus V_{16}$
E_8	28	$V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}$

It is immediately clear that A has only one structure of an algebra for type A_n and no more than one structure of a semisimple algebra for types D_n, T_n .

By Lemma 7 the object $A = V_0 \oplus V_l$ of \mathcal{C}_l has a unique structure of a semisimple associative algebra for even l and has no such structure for odd l . In particular, the module category of type T_n does not exist and the module category of type D_n does exist and is unique.

To prove uniqueness for types E_6, E_7 we will use the following

Lemma 8. Let \mathcal{C} be a rigid monoidal category and X be an irreducible object of \mathcal{C} . Assume that $\text{Hom}(X \otimes X, X)$ is one dimensional. Then $A = \mathbf{1} \oplus X$ has at most one structure of a semisimple algebra in \mathcal{C} .

Proof is a word by word repetition of the argument in [24], page 24 Type E_6 . \square

The Lemma implies immediately that the algebra $A = V_0 \oplus V_6$ of type E_6 is unique. The algebra of type E_7 contains a subalgebra $A' = V_0 \oplus V_{16}$. So we can consider this algebra as an algebra in the monoidal category of A' -bimodules, $A = A' \oplus X$ where X is A' -bimodule and $X = V_8$ as an object of \mathcal{C}_{16} . One verifies easily along the lines of [24] Section 8 that X has four possible structures of an A' -bimodule and is irreducible as an A' -bimodule. For two of these structures $X \otimes_{A'} X$ does not contain A' as a direct summand and so these bimodules cannot appear in a semisimple algebra A . Two other bimodule structures are permuted by the automorphism of the algebra A' which is 1 on V_0 and (-1) on V_{16} so it is enough to consider only one such structure. Now we can apply the Lemma above to get the uniqueness of an algebra of type E_7 .

Now consider the case of the algebra A of type E_8 . In this case again $A = A' \oplus X$ where $A' = V_0 \oplus V_{28}$ is a subalgebra and $X = V_{10} \oplus V_{18}$ is an A' -bimodule. One shows that there are two possible structures $X^{(1)}, X^{(2)}$ of an A' -bimodule on X : $X^{(1)}$ comes from $X = \alpha^+(V_{10})$ (where the α -induction is taken with respect to the category of A' -bimodules) and the right A' -action of A' on $X^{(2)}$ differs from that on $X^{(1)}$ by an automorphism of A' which is 1 on V_0 and -1 on V_{28} (the left A' -actions on $X^{(1)}$ and $X^{(2)}$ are assumed to be the same). One verifies easily that $X^{(1)} \otimes_{A'} X^{(1)} = X^{(2)} \otimes_{A'} X^{(2)}$ as A' -bimodules and moreover this tensor product contains $X^{(1)}$ with multiplicity 2 as a direct summand and does not contain $X^{(2)}$. This implies that in the algebra A one has $X = X^{(1)}$ (otherwise the product $X \times X \rightarrow X$ is zero which is possible iff $X \otimes_{A'} X = A'$) and A' lies in the “two sided” center (intersection of B^+ and B^-) of A . Now one shows easily that the multiplication in A is commutative: the maps $V_{10} \otimes V_{10} \rightarrow A$ and $V_{18} \otimes V_{18} \rightarrow A$ commute with braiding by Lemma 7.5 of [24] and the maps $V_{10} \otimes V_{18} \rightarrow A$ and $V_{18} \otimes V_{10} \rightarrow A$ are permuted by the braiding thanks to the associativity since V_{18} is the image of the multiplication of V_{10} and V_{28} . Finally, the uniqueness of the commutative algebra $A = V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}$ was shown in [24]. Note that using the known structure of the modular invariant of type E_8 and Claim 5 above one deduces the commutativity of the algebra of type E_8 immediately.

The Theorem is proved. \square

Remark 14. (i) The Theorem above is not new (but probably our way to state it is new). Unfortunately the history is a bit complicated and I don't know who should get credit for it. This Theorem was undoubtedly known to physicists for some time; I believe that A. Ocneanu was the first who translated this result into mathematical language (in a context of the subfactors theory). Unfortunately his results are difficult to understand for a nonexpert in Operator Algebras; one of the purposes of this paper is to make these remarkable results accessible to a student with standard background in algebra. At least a big portion of Theorem 6 is contained in [6], in particular the subfactor theory construction of the module category of type E_7 , see [6] Appendix (unfortunately I don't understand this construction; in fact this paper grew up from my attempts to understand it). The idea of using the conformal

inclusions to construct module categories was first translated into mathematical language by F. Xu, see [37].

(ii) Using the Theorem above and Remark 13 one finds that the indecomposable module categories over \mathcal{C}_l^{tw} are classified by the Dynkin diagrams of types $A_n, D_n, T_n, E_6, E_7, E_8$. This gives some explanation why it is difficult to rule out tadpoles T_n using just combinatorial methods.

(iii) The situation with module categories (or modular invariants) of type D_n has a vast generalization known under the name of simple currents. This can be summarized as follows: let \mathcal{C} be a monoidal category such that each irreducible object of \mathcal{C} is invertible. Such a category is completely determined by the group G of isomorphism classes of irreducible objects and by the class $\omega \in H^3(G, \mathbb{C}^*)$, see e.g. [33, 35]. The object $A = \bigoplus_{g \in G} X_g$ where X_g is a representative of the isomorphism class g has a structure of a semisimple algebra if and only if the class ω is trivial. In this case the possible structures of a semisimple algebra on A are classified by $H^2(G, \mathbb{C}^*)$ (“discrete torsion”), see e.g. [14].

(iv) A lot of explicit information on the module categories above is available in the literature. For example the structure of the categories \mathcal{C}^* (e.g. the Grothendieck ring) is known in all cases thanks to the work of A. Ocneanu. This information is usually presented in the form of the “Ocneanu graphs”, see the beautiful pictures e.g. in [30, 31, 32]. Moreover, A. Ocneanu calculated in all cases categories $\text{Func}_{\mathcal{C}_l}(\mathcal{M}_1, \mathcal{M}_2)$ where $\mathcal{M}_1, \mathcal{M}_2$ are possibly different module categories over \mathcal{C}_l , see *loc. cit.* For each irreducible object $F \in \mathcal{C}^*$ one associates the “twisted partition function” which is a matrix $a_{ij} := \dim \text{Hom}(\mathbf{1}, \alpha^+(V_i) \otimes F \otimes \alpha^-(V_j))$ where $i, j \in 0, 1, \dots, l$. The paper [9] contains the tables of all twisted partition functions.

(v) Of course there is an obvious problem to generalize the classification above to the other simple Lie algebras. One can find in [18] a good account of the related combinatorics. I believe that A. Ocneanu solved the corresponding problem for $\widehat{sl}(3)$ and $\widehat{sl}(4)$, see e.g. his announcement [31]. It would be extremely interesting to see the details of his work (using our methods we are probably able to reprove part of his results, but I don’t know how to construct module categories of types $\mathcal{A}_n^*, \mathcal{D}_n^*$ over $\widehat{sl}(3)$).

(vi) Let X be a Dynkin diagram of type A, D, E and let \mathcal{M} be the corresponding module category over \mathcal{C}_l . Let $I = \{0, 1, \dots, l\}$ be the set labeling irreducible objects of \mathcal{C}_l and let \mathfrak{A} be the set labeling irreducible objects of \mathcal{M} . For any $a \in \mathfrak{A}$ let M_a be the corresponding object. For $i \in I, a, b \in \mathfrak{A}$ consider the vector space $W_{ab}^i := \text{Hom}(V_i \otimes M_a, M_b)$ (in the terminology of Ocneanu, this is the space of essential paths from a to b of length i , see [30]). Using the canonical morphisms $V_{i+j} \rightarrow V_i \otimes V_j$ (where $V_{i+j} = 0$ if $i + j > l$) one defines a multiplication $W_{ab}^i \otimes W_{bc}^j \rightarrow W_{ac}^{i+j}$ which makes the direct sum $\bigoplus_{a,b,i} W_{ab}^i$ into an associative algebra. One recognizes immediately that this algebra is exactly the Gelfand-Ponomarev preprojective algebra associated to X , see e.g. [26]. So this gives an amazing possibility that the module categories over \mathcal{C}_l are related with the quiver varieties. I don’t know if it is possible to pursue this relation further. Applying a similar construction to the module categories over $\widehat{sl}(3), \widehat{sl}(4)$ etc one gets a vast generalization of preprojective algebras and perhaps it would be interesting to study these objects.

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