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On commutators and derivations in rings

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Abstract

We consider the problem when the product of certain higher commutators arising from a fixed element in a ring lies in the ideal generated by some power of this element. The result which we obtain is applied to the study of (generalized) derivations in rings and (Banach) algebras.

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1. Introduction

Let $\mathcal{A}$ be a ring. As usual we denote the commutator $ab - ba$ of elements $a, b \in \mathcal{A}$ by $[a, b]$. In our main result (Theorem 2.1) we discover some elementary properties of higher commutators of the form $[a, [a, [a, \dots, [a, b] \dots]]]$. A simplified version of its most interesting part reads as follows.

Theorem 1.1. *Let $\mathcal{A}$ be an algebra over a field F with* $\operatorname{char}(F) = 0$. *Let $a, b_0, c_0 \in \mathcal{A}$ and set $b_i = [a, b_{i-1}]$, $c_i = [a, c_{i-1}]$ for every $i \geqslant 1$. Suppose that $[b_i, c_j] = 0$ for all $i, j \geqslant 0$. Let m_1, m_2 be odd positive integers and let $n = (m_1 + m_2 + 2)/2$. Then $b_{m_1} c_{m_2}$ lies in the ideal of $\mathcal{A}$ generated by a^n.*

Since $b_{m_1} = \sum_{j=0}^{m_1} (-1)^j \binom{m_1}{j} a^{m_1 - j} b_0 a^j$ it is clear that b_{m_1} lies in the ideal of $\mathcal{A}$ generated by $a^{(m_1+1)/2}$, and similarly, c_{m_2} lies in the ideal generated by $a^{(m_2+1)/2}$.

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Theorem 1.1 tells us that $b_{m_1} \cdot c_{m_2}$ lies in the ideal generated by the product $a^{(m_1+1)/2} \cdot a^{(m_2+1)/2}$ of these two generators.

Theorem 2.1 gives a much more detailed information. In particular, it considers the situation when $\mathcal{A}$ is an arbitrary ring.

When m_1 and m_2 are even this result fails. However, in this case we have a different conclusion (see the last sentence of Theorem 2.1), though from the point of view of applications apparently a less useful one.

The assumption on the commutativity of b_i and c_j seems to be a rather strong one, but there is a special instance where it is automatically fulfilled. It concerns derivations of algebras. Let $\mathcal{A}$ be an algebra and let $\mathcal{L}(\mathcal{A})$ be the algebra of all linear operators from $\mathcal{A}$ into $\mathcal{A}$. For every $x, y \in \mathcal{A}$ we define $L_x, R_y \in \mathcal{L}(\mathcal{A})$ by $L_x(a) = xa$, $R_y(a) = ay$. Obviously, L_x and R_y commute for any x and y. A derivation of $\mathcal{A}$ can be defined as an operator $D \in \mathcal{L}(\mathcal{A})$ such that $L_{D(x)} = [D, L_x]$ for every $x \in \mathcal{A}$, or equivalently, $R_{D(y)} = [D, R_y]$ for every $y \in \mathcal{A}$. Accordingly, $L_{D^i(x)} = [D, L_{D^{i-1}(x)}]$ and $R_{D^i(y)} = [D, R_{D^{i-1}(y)}]$ for every $i \geqslant 1$. Letting D, L_x and R_y to play the roles of a, b_0 and c_0 respectively (and of course $\mathcal{L}(\mathcal{A})$ playing the role of $\mathcal{A}$), we immediately get the following corollary to Theorem 1.1.

Corollary 1.2. *Let $\mathcal{A}$ be an algebra over a field F with $\operatorname{char}(F) = 0$. Let D be a derivation of $\mathcal{A}$ and let $x, y \in \mathcal{A}$. Further, let m_1, m_2 be odd positive integers and let $n = (m_1 + m_2 + 2)/2$. Then the operator $L_{D^{m_1}(x)} R_{D^{m_2}(y)}$ lies in the ideal of $\mathcal{L}(\mathcal{A})$ generated by D^n.*

A more precise version of Corollary 1.2 (Corollary 3.1) is given at the beginning of Section 3 where one can also find some concrete formulae that illustrate our results. Also, a similar result is stated for the so-called generalized derivations for which our main result is also applicable.

Section 2 is devoted to our main result, and Section 3 to its various applications: we shall generalize and unify various existing results concerning (and related to) nilpotency of derivations, complete and extend a recent result of the first two authors on derivations certain of whose powers have finite rank, and finally indicate (the full treatment will be done in another paper) possible applications to the study of derivations on Banach algebras.

All applications of our main result that we have found so far treat (generalized) derivations, and moreover only in the context of noncommutative rings and algebras. In view of the elementary nature of this result, however, we hope that it might turn out to be applicable also in some other areas.

2. The main result

Let us fix our notation. Throughout this section, $\mathcal{A}$ will be an arbitrary associative ring (possibly without an identity element). We write $x \circ y = xy + yx$ for any $x, y \in \mathcal{A}$. Note that $[y, y \circ x] = y \circ [y, x] = [y^2, x]$; we will use this without comment in the sequel.

Let a, b_0, c_0 be fixed elements in $\mathcal{A}$, and set

$$b_i = [a, b_{i-1}], \quad c_i = [a, c_{i-1}] \quad \text{for any } i \geqslant 1.$$

So, for example, $c_1 = [a, c_0]$, $c_2 = [a, [a, c_0]]$, $c_3 = [a, [a, [a, c_0]]]$, etc. It is well known and easy to see that

$$c_i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} a^{i-j} c_0 a^j \quad \text{for all } i \geqslant 1. \tag{1}$$

Next, $n \geqslant 2$ will be a fixed integer. For any $m = 1, \dots, n-1$ we introduce the $m \times m$ matrix

$$A_m = \begin{pmatrix} \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{m} \\ \binom{n+1}{2} & \binom{n+1}{3} & \cdots & \binom{n+1}{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m-1}{m} & \binom{n+m-1}{m+1} & \cdots & \binom{n+m-1}{2m-1} \end{pmatrix}$$

and set $\delta_m = \det(A_m)$. We also set $\delta_0 = 1$ and define

$$\Delta_m = \delta_{n-m}\delta_{n-m+1}\dots\delta_{n-1}, \qquad m = 1, \dots, n.$$

Finally, by $\mathcal{G}$ we denote the additive subgroup of $\mathcal{A}$ generated by all elements of the form $a^{n+k} b_i c_j$, $b_i a^{n+k} c_j$, $c_j a^{n+k} b_i$ and $b_i c_j a^{n+k}$ where i, j, k are nonnegative integers with $i + j + k = n - 2$.

We are now in a position to state our main result.

Theorem 2.1. *Suppose that* $[b_i, c_j] = 0$ *whenever* $i \geqslant 0$, $0 \leqslant j \leqslant n-2$, *and* $0 \leqslant i + j \leqslant 2n - 2$. *Let* $m \leqslant n$ *and* $k \leqslant n-1$ *be positive integers.*

(i) *If* m *is even then* $\Delta_m b_{k+n-m-1}[a^{n-k}, a \circ c_{m-2}] \in \mathcal{G}$.
(ii) *If* m *is odd then* $\Delta_m b_{k+n-m-1}[a^{n-k}, c_{m-1}] \in \mathcal{G}$.

In particular, $\Delta_m b_{2n-m-2}(a \circ c_{m-1}) \in \mathcal{G}$ *if* m *is even, and* $\Delta_m b_{2n-m-2} c_m \in \mathcal{G}$ *if* m *is odd.*

The last sentence is just another way of stating (i) and (ii) for $k = n-1$. Namely, if $k = n-1$ then $[a^{n-k}, a \circ c_{m-2}] = [a, a \circ c_{m-2}] = a \circ [a, c_{m-2}] = a \circ c_{m-1}$ and $[a^{n-k}, c_{m-1}] = [a, c_{m-1}] = c_m$.

Theorem 2.1 would be meaningless without some additional information on the integers Δ_m. For our purposes the following result is sufficient.

Lemma 2.2. *A prime number* $p \geqslant 2n-1$ *does not divide* Δ_m *for every* $m = 1, \dots, n$.

Proof. We fix m with $1 \leqslant m \leqslant n-1$. It suffices to show that p does not divide δ_m.

By $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ we denote the rows of the matrix A_m. Let B_m be the matrix whose k-th row is equal to $\sum_{i=1}^{k}(-1)^{k-i}\binom{k-1}{i-1}\mathbf{a}_i$. Of course B_m has the same determinant as A_m, i.e., $\det(B_m) = \delta_m$. Note that $(B_m)_{k,j}$, the (k, j)-entry of B_m, is equal to

$$\sum_{i=1}^{k}(-1)^{k-i}\binom{k-1}{i-1}\binom{n+i-1}{j+i-1} = \sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}\binom{n+i}{j+i}.$$

This sum can be presented in a simple way. Indeed, since the coefficient of $(1+X)^s$ at X^t is $\binom{s}{t}$ it follows by comparing the coefficients at X^{k+j-1} in

$$(1+X)^n = \bigl(-X + (1+X)\bigr)^{k-1}(1+X)^n = \sum_{i=0}^{k-1}\binom{k-1}{i}(-X)^{k-1-i}(1+X)^{n+i}$$

that $(B_m)_{k,j} = \binom{n}{j+k-1}$ where it should be understood that $\binom{n}{s} = 0$ if $s > n$.

For any integer r we shall write $\bar{r} = r + p\mathbb{Z} \in \mathbb{Z}_p$. Let $\overline{B}_m \in M_m(\mathbb{Z}_p)$ be the matrix whose (k, j)-entry is equal to

$$\overline{(B_m)_{k,j}} = \overline{\binom{n}{j+k-1}}.$$

The lemma will be proved by showing that $\det(\overline{B}_m) \neq 0$. Equivalently, we have to show that the rows $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ of $\overline{B}_m$ are linearly independent. So assume that there are $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{Z}_p$ such that

$$\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_m\mathbf{b}_m = \mathbf{0} \in \mathbb{Z}_p^m. \tag{2}$$

Define $f(X) \in \mathbb{Z}_p[X]$ by

$$f(X) = (1+X)^n\bigl(\alpha_1X^{m-1} + \alpha_2X^{m-2} + \cdots + \alpha_m\bigr).$$

Writing $f(X) = \sum_{i=0}^{n+m-1}\beta_iX^i$ we see that $\beta_{n+m-1} = \alpha_1$, $\beta_{n+m-2} = \binom{n}{n-1}\alpha_1 + \alpha_2$, etc. The key observation, however, is that (2) yields $\beta_i = 0$ whenever $m \leqslant i \leqslant 2m-1$. That is,

$$f^{(m)}(0) = f^{(m+1)}(0) = \cdots = f^{(2m-1)}(0) = 0. \tag{3}$$

Note that $f^{(m)}(X) = (1+X)^{n-m}h(X)$ for some $h(X) \in \mathbb{Z}_p[X]$ of degree at most $m-1$. From (3) it follows that $h(0) = h'(0) = \cdots = h^{(m-1)}(0) = 0$ which shows that $h(X)$ is actually 0. Consequently, $f^{(m)}(X) = 0$, and hence $\overline{(n+m-1)(n+m-2)\cdots n}\beta_{n+m-1} = 0$. Since $p > n+m-1$ we see that $\overline{(n+m-1)(n+m-2)\cdots n} \neq 0$, which in turn implies that $\beta_{n+m-1} = 0$. That is, $\alpha_1 = 0$. The same argument then shows that $\alpha_2 = 0$, $\alpha_3 = 0$, etc. □

The proof of Theorem 2.1 will be given at the end of this section. First we shall derive some other properties of b_m and c_m.

Lemma 2.3. *For all* $k \geqslant 1$ *and* $m \geqslant 0$ *we have* $a^k b_m = \sum_{j=0}^{k} \binom{k}{j} b_{m+j} a^{k-j}$.

Proof. Using $a^{k+1} b_m = [a, a^k b_m] + (a^k b_m)a$ one easily proves the lemma by induction on k. □

By $\langle x_1, x_2, \ldots, x_m \rangle$ we denote the additive subgroup of $\mathcal{A}$ generated by $x_1, x_2, \ldots, x_m \in \mathcal{A}$.

Lemma 2.4. *Let* t *be a positive integer, and set*

$$u_i = \begin{cases} [a^i, c_{2t+1-i}] & \text{if } i \text{ is odd,} \\ [a^i, a \circ c_{2t-i}] & \text{if } i \text{ is even,} \end{cases} \qquad i = 1, 2, \ldots, 2t+1.$$

Then $u_1, u_2, \ldots, u_t \in \langle u_{t+1}, u_{t+2}, \ldots, u_{2t+1} \rangle$.

Proof. Let $w_s = a^{2t+1-s} c_0 a^s - a^s c_0 a^{2t+1-s}$, $0 \leqslant s \leqslant t$. We claim that

$$w_s \in \langle u_{t+1}, u_{t+2}, \ldots, u_{2t+1} \rangle, \quad 0 \leqslant s \leqslant t. \tag{4}$$

For $s = 0$ this is trivial since $w_0 = u_{2t+1}$. So we may assume that $s > 0$ and that (4) holds true for any nonnegative integer smaller than s. Suppose first that s is even. Using (1) we have

$$\begin{aligned} u_{2t+1-s} &= \left[a^{2t+1-s}, c_s\right] \\ &= \left[a^{2t+1-s}, \sum_{j=0}^{s} (-1)^j \binom{s}{j} a^{s-j} c_0 a^j\right] \\ &= \sum_{j=0}^{s} (-1)^j \binom{s}{j} a^{2t+1-j} c_0 a^j - \sum_{j=0}^{s} (-1)^j \binom{s}{j} a^{s-j} c_0 a^{2t+1-s+j} \\ &= \sum_{j=0}^{s} (-1)^j \binom{s}{j} a^{2t+1-j} c_0 a^j - \sum_{j=0}^{s} (-1)^j \binom{s}{j} a^{j} c_0 a^{2t+1-j} \\ &= \left(\sum_{j=0}^{s-1} (-1)^j \binom{s}{j} w_j\right) + w_s, \end{aligned}$$

which clearly yields (4). Assuming that s is odd we have

$$\begin{aligned} u_{2t+1-s} &= \left[a^{2t+1-s}, a \circ c_{s-1}\right] \\ &= \left[a^{2t+1-s}, a \circ \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} a^{s-1-j} c_0 a^j\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\bigl(a^{2t+1-j}c_0a^j + a^{2t-j}c_0a^{j+1}\bigr)\\
&\quad - \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\bigl(a^{s-j}c_0a^{2t+1-s+j} + a^{s-1-j}c_0a^{2t+2-s+j}\bigr)\\
&= \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\bigl(a^{2t+1-j}c_0a^j + a^{2t-j}c_0a^{j+1}\bigr)\\
&\quad - \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\bigl(a^{j+1}c_0a^{2t-j} + a^jc_0a^{2t+1-j}\bigr)\\
&= \left(\sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}w_j\right) + \left(\sum_{j=0}^{s-2}(-1)^j\binom{s-1}{j}w_{j+1}\right) + w_s,
\end{aligned}
$$

and so (4) holds in this case too.

In view of (4) it suffices to prove that $u_i \in \langle w_0, w_1, \ldots, w_t\rangle$, $1 \leqslant i \leqslant t$. For convenience we define $w_{t+j} = -w_{t+1-j}$ for $j = 1, \ldots, t+1$. First assume that i is odd. Then

$$
\begin{aligned}
u_i &= \bigl[a^i, c_{2t+1-i}\bigr] = \left[a^i, \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}a^{2t+1-i-j}c_0a^j\right]\\
&= \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}a^{2t+1-j}c_0a^j\\
&\quad - \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}a^{2t+1-i-j}c_0a^{i+j}\\
&= \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}a^{2t+1-j}c_0a^j - \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}a^jc_0a^{2t+1-j}\\
&= \sum_{j=0}^{2t+1-i}(-1)^j\binom{2t+1-i}{j}w_j \in \langle w_0, w_1, \ldots, w_t\rangle.
\end{aligned}
$$

Now assume that i is even. Then

$$
u_i = \bigl[a^i, a \circ c_{2t-i}\bigr] = \left[a^i, a \circ \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}a^{2t-i-j}c_0a^j\right]
$$

$$\begin{aligned}
&= \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}\left(a^{2t+1-j}c_0a^j + a^{2t-j}c_0a^{j+1}\right)\\
&\quad - \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}\left(a^{2t+1-i-j}c_0a^{i+j} + a^{2t-i-j}c_0a^{i+j+1}\right)\\
&= \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}\left(a^{2t+1-j}c_0a^j + a^{2t-j}c_0a^{j+1}\right)\\
&\quad - \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}\left(a^{j+1}c_0a^{2t-j} + a^jc_0a^{2t+1-j}\right)\\
&= \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}w_j + \sum_{j=0}^{2t-i}(-1)^j\binom{2t-i}{j}w_{j+1}\\
&\in \langle w_0, w_1, \ldots, w_t\rangle. \qquad \square
\end{aligned}$$

Lemma 2.5. *Let t be a positive integer, and set*

$$v_i = \begin{cases} [a^i, a\circ c_{2t+1-i}] & \text{if } i \text{ is odd},\\ [a^i, c_{2t+2-i}] & \text{if } i \text{ is even}, \end{cases} \qquad i = 2, 3, \ldots, 2t+2.$$

Then $v_2, v_3, \ldots, v_{t+1} \in \langle v_{t+2}, v_{t+3}, \ldots, v_{2t+2}\rangle$.

Proof. We define $z_s = a^{2t+2-s}c_0a^s - a^sc_0a^{2t+2-s}$, $0 \leqslant s \leqslant t$. In the first step of the proof we show that $z_s \in \langle v_{t+2}, v_{t+3}, \ldots, v_{2t+2}\rangle$, $0 \leqslant s \leqslant t$, and in the second step we show that $v_i \in \langle z_0, z_1, \ldots, z_t\rangle$, $2 \leqslant i \leqslant t+1$. The proof is similar as that of the previous lemma, so we omit details. $\square$

Proof of Theorem 2.1. Recall that $\mathcal{G}$ is the additive subgroup of $\mathcal{A}$ generated by elements $a^{n+k}b_ic_j$, $b_ia^{n+k}c_j$, $c_ja^{n+k}b_i$ and $b_ic_ja^{n+k}$ where i, j, k are nonnegative integers with $i+j+k = n-2$. We shall write $x \equiv 0$ to denote that $x \in \mathcal{G}$.

Let $0 \leqslant s \leqslant l$ be integers. Using Lemma 2.3 we see that

$$\begin{aligned}
&a^{n+s}b_{l-s} - \sum_{j=0}^{s}\binom{n+s}{j}b_{l-s+j}a^{n+s-j}\\
&\quad = \sum_{j=1+s}^{n+s}\binom{n+s}{j}b_{l-s+j}a^{n+s-j} = \sum_{k=1}^{n}\binom{n+s}{k+s}b_{k+l}a^{n-k}.
\end{aligned} \tag{5}$$

Now assume that $l \leqslant n-2$. Then, according to our assumption, $[b_{k+l}, c_{n-l-2}] = 0$ for all $k = 1, \ldots, n$. Using this together with (5) we then get

$$\sum_{k=1}^{n-1}\binom{n+s}{k+s}b_{k+l}\bigl[a^{n-k},c_{n-l-2}\bigr]=\Biggl[\sum_{k=1}^{n}\binom{n+s}{k+s}b_{k+l}a^{n-k},c_{n-l-2}\Biggr]$$
$$=\Biggl[a^{n+s}b_{l-s}-\sum_{j=0}^{s}\binom{n+s}{j}b_{l-s+j}a^{n+s-j},c_{n-l-2}\Biggr],$$

which clearly implies that

$$\sum_{k=1}^{n-1}\binom{n+s}{k+s}b_{k+l}\bigl[a^{n-k},c_{n-l-2}\bigr]\equiv 0,\quad 0\leqslant s\leqslant l\leqslant n-2. \tag{6}$$

This is our first key relation. Let us now derive the second one, under the additional assumption that $n\geqslant 3$. Let $0\leqslant l\leqslant n-3$. Note that then $[b_{k+l},c_{n-l-3}]=0$ for all $k=1,\dots,n$, and so using (5) we get

$$\sum_{k=1}^{n-1}\binom{n+s}{k+s}b_{k+l}\bigl[a^{n-k},c_{n-l-3}\bigr]a$$
$$=\Biggl[\sum_{k=1}^{n}\binom{n+s}{k+s}b_{k+l}a^{n-k},c_{n-l-3}\Biggr]a$$
$$=\Biggl[a^{n+s}b_{l-s}-\sum_{j=0}^{s}\binom{n+s}{j}b_{l-s+j}a^{n+s-j},c_{n-l-3}\Biggr]a$$
$$=a^{n+s}b_{l-s}c_{n-l-3}a-\sum_{j=0}^{s}\binom{n+s}{j}b_{l-s+j}a^{n+s-j}c_{n-l-3}a$$
$$-c_{n-l-3}a^{n+s}b_{l-s}a+\sum_{j=0}^{s}\binom{n+s}{j}c_{n-l-3}b_{l-s+j}a^{n+s-j+1}.$$

In view of $b_ma=ab_m-b_{m+1}$ and $c_ma=ac_m-c_{m+1}$ it follows easily that each of the terms on the right hand side lies in $\mathcal{G}$. For example,

$$a^{n+s}b_{l-s}c_{n-l-3}a=a^{n+s}b_{l-s}ac_{n-l-3}-a^{n+s}b_{l-s}c_{n-l-2}$$
$$=a^{n+s+1}b_{l-s}c_{n-l-3}-a^{n+s}b_{l-s+1}c_{n-l-3}-a^{n+s}b_{l-s}c_{n-l-2}\equiv 0,$$

while for other terms the argument is even shorter. Accordingly,

$$\sum_{k=1}^{n-1}\binom{n+s}{k+s}b_{k+l}\bigl[a^{n-k},c_{n-l-3}\bigr]a\equiv 0,$$

which together with (6) yields

$$0 \equiv \sum_{k=1}^{n-1} \binom{n+s}{k+s} b_{k+l}\big[a^{n-k}, c_{n-l-2}\big] + 2\sum_{k=1}^{n-1} \binom{n+s}{k+s} b_{k+l}\big[a^{n-k}, c_{n-l-3}\big]a$$
$$= \sum_{k=1}^{n-1} \binom{n+s}{k+s} b_{k+l}\big(\big[a^{n-k}, [a, c_{n-l-3}]\big] + 2\big[a^{n-k}, c_{n-l-3}\big]a\big).$$

Noting that $[a^{n-k}, [a, c_{n-l-3}]] + 2[a^{n-k}, c_{n-l-3}]a = [a^{n-k}, a \circ c_{n-l-3}]$, we thus get our second key relation:

$$\sum_{k=1}^{n-1} \binom{n+s}{k+s} b_{k+l}\big[a^{n-k}, a \circ c_{n-l-3}\big] \equiv 0, \quad 0 \leqslant s \leqslant l \leqslant n-3. \tag{7}$$

We shall prove the theorem by induction on m. The case when $m = 1$ is now easy to handle. Just set $l = n - 2$ in (6) and note that the relations obtained can be expressed through the matrix A_{n-1}, i.e.,

$$\begin{pmatrix} \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ \binom{n+1}{2} & \binom{n+1}{3} & \cdots & \binom{n+1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2n-2}{n-1} & \binom{2n-2}{n} & \cdots & \binom{2n-2}{2n-3} \end{pmatrix} \begin{pmatrix} b_{n-1}[a^{n-1}, c_0] \\ b_n[a^{n-2}, c_0] \\ \vdots \\ b_{2n-3}[a, c_0] \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

Multiplying this relation by the adjoint of A_{n-1} we obtain $\delta_{n-1}b_{n-1}[a^{n-1}, c_0] \equiv 0$, $\delta_{n-1}b_n[a^{n-2}, c_0] \equiv 0, \ldots, \delta_{n-1}b_{2n-3}[a, c_0] \equiv 0$, which is exactly the desired conclusion for $m = 1$.

So we may assume that $m > 1$. In the special case where $n = m = 2$, $\mathcal{G}$ is generated by elements $a^2b_0c_0$, $b_0a^2c_0$, $c_0a^2b_0$ and $b_0c_0a^2$, and so $b_0[a, a \circ c_0] = b_0a^2c_0 - b_0c_0a^2 \in \mathcal{G}$, as desired. The theorem is thereby proved for $n = 2$. So let $n \geqslant 3$. Further, we may assume that the theorem is true for every positive integer smaller than m, that is,

$$\Delta_{m'} b_{k'+n-m'-1}\big[a^{n-k'}, a \circ c_{m'-2}\big] \equiv 0, \quad m' \text{ even},\ 1 \leqslant m' \leqslant m-1,\ 1 \leqslant k' \leqslant n-1, \tag{8}$$

and

$$\Delta_{m'} b_{k'+n-m'-1}\big[a^{n-k'}, c_{m'-1}\big] \equiv 0, \quad m' \text{ odd},\ 1 \leqslant m' \leqslant m-1,\ 1 \leqslant k' \leqslant n-1. \tag{9}$$

(i) Let us first consider the case when m is even. Our first goal is to prove that

$$\Delta_{m-1} b_{k+n-m-1}\big[a^{n-k}, a \circ c_{m-2}\big] \equiv 0 \quad \text{when } n-m+1 \leqslant k \leqslant n-1. \tag{10}$$

First of all, setting $k' = n - 2$ and $m' = m - 1$ (which is indeed odd) in (9) we get $\Delta_{m-1}b_{2n-m-2}[a^2, c_{m-2}] \equiv 0$. Since $[a^2, c_{m-2}] = [a, a \circ c_{m-2}]$ this shows that (10) holds true for $k = n - 1$. So we may assume that $k \leqslant n - 2$. We have $m = 2p$ with $p \geqslant 1$ and $2p \leqslant n$. Let us fix k such that $n - m + 1 \leqslant k \leqslant n - 2$. We have to treat separately two cases, when $n - k$ is odd and when it is even.

Case 1. Assume first that $n-k$ is odd. So we have $n-k=2q+1$ and note that $1\leqslant q<p$. Set $t=p+q-1$. Note that $n-k=2q+1\leqslant p+q=t+1$. Therefore, adopting the notation of Lemma 2.5 we have

$$v_{n-k}=\left[a^{n-k},a\circ c_{m-2}\right]\in\langle v_{t+2},v_{t+3},\ldots,v_{2t+2}\rangle. \tag{11}$$

Again we have to divide the proof into two parts.

Subcase 1a. We first treat the case when $p+q$ is an odd number. Then (11) can be written as

$$\left[a^{n-k},a\circ c_{m-2}\right]\in\left\langle\left[a^{p+q+1},c_{p+q-1}\right],\left[a^{p+q+2},a\circ c_{p+q-3}\right],\left[a^{p+q+3},c_{p+q-3}\right],\ldots,\left[a^{2p+2q-1},a\circ c_0\right],\left[a^{2p+2q},c_0\right]\right\rangle.$$

Since $k+n-m-1=2n-(n-k)-m-1=2n-(2q+1)-2p-1=2(n-p-q-1)$, (10) will be proved by showing that

$$\Delta_{m-1}b_{2(n-p-q-1)}\left[a^{p+q+2r-1},c_{p+q-2r+1}\right]\equiv 0,\quad 1\leqslant r\leqslant\frac{p+q+1}{2}, \tag{12}$$

and

$$\Delta_{m-1}b_{2(n-p-q-1)}\left[a^{p+q+2r},a\circ c_{p+q-2r-1}\right]\equiv 0,\quad 1\leqslant r\leqslant\frac{p+q-1}{2}. \tag{13}$$

First we prove (12). If $p+q+2r-1\geqslant n$ then (12) trivially holds, so we may assume that $p+q+2r-1<n$. Therefore $k'=n-p-q-2r+1\geqslant 1$. Clearly $k'\leqslant n-1$. Further, $m'=p+q-2r+2$ is odd (since $p+q$ is odd), $m'\geqslant 1$ (since $r\leqslant(p+q+1)/2$) and $m'\leqslant m-1$ (since $q<p$ and $r\geqslant 1$). Therefore $\Delta_{m'}b_{k'+n-m'-1}[a^{n-k'},c_{m'-1}]\equiv 0$ by (9). Since $k'+n-m'-1=2(n-p-q-1)$, $n-k'=p+q+2r-1$, $m'-1=p+q-2r+1$, and $\Delta_{m'}$ divides Δ_{m-1} (since $m'\leqslant m-1$), this proves (12).

The proof of (13) is similar. Now we may assume that $p+q+2r<n$ and so $1\leqslant k'=n-p-q-2r\leqslant n-1$. Further, note that $m'=p+q-2r+1$ is even and $1\leqslant m'\leqslant m-1$. Therefore (13) follows from (8). This completes the proof for this case.

Subcase 1b. Now assume that $p+q$ is an even number, so that (11) can be rewritten as

$$\left[a^{n-k},a\circ c_{m-2}\right]\in\left\langle\left[a^{p+q+1},a\circ c_{p+q-2}\right],\left[a^{p+q+2},c_{p+q-2}\right],\left[a^{p+q+3},a\circ c_{p+q-4}\right],\ldots,\left[a^{2p+2q-1},a\circ c_0\right],\left[a^{2p+2q},c_0\right]\right\rangle.$$

Therefore (10) will follow from

$$\Delta_{m-1}b_{2(n-p-q-1)}\left[a^{p+q+2r-1},a\circ c_{p+q-2r}\right]\equiv 0,\quad 1\leqslant r\leqslant\frac{p+q}{2}, \tag{14}$$

and

$$\Delta_{m-1}b_{2(n-p-q-1)}\big[a^{p+q+2r}, c_{p+q-2r}\big] \equiv 0, \quad 1 \leqslant r \leqslant \frac{p+q}{2}. \tag{15}$$

In order to prove (14) we may assume that $p+q+2r-1<n$. Note that then (14) follows from (8) by choosing $k'=n-p-q-2r+1$ and $m'=p+q-2r+2$. Similarly, we may assume that $p+q+2r<n$ and then (9) with $k'=n-p-q-2r$ and $m'=p+q-2r+1$ implies (15).

Case 2. Now suppose that $n-k$ is even, i.e., $n-k=2q$ with $1 \leqslant q < p$. Again we set $t=p+q-1$. Then $n-k=2q \leqslant t$, and so Lemma 2.4 tells us that

$$u_{n-k}=\big[a^{n-k}, a \circ c_{m-2}\big] \in \langle u_{t+1}, u_{t+2}, \ldots, u_{2t+1}\rangle. \tag{16}$$

Subcase 2a. Suppose that $p+q$ is odd. Then (16) reads as

$$\big[a^{n-k}, a \circ c_{m-2}\big] \in \big\langle \big[a^{p+q}, c_{p+q-1}\big], \big[a^{p+q+1}, a \circ c_{p+q-3}\big], \big[a^{p+q+2}, c_{p+q-3}\big], \\ \ldots, \big[a^{2p+2q-2}, a \circ c_0\big], \big[a^{2p+2q-1}, c_0\big]\big\rangle.$$

Thus, to prove (10) it is enough to show that

$$\Delta_{m-1}b_{2(n-p-q)-1}\big[a^{p+q+2r-2}, c_{p+q-2r+1}\big] \equiv 0, \quad 1 \leqslant r \leqslant \frac{p+q+1}{2}, \tag{17}$$

and

$$\Delta_{m-1}b_{2(n-p-q)-1}\big[a^{p+q+2r-1}, a \circ c_{p+q-2r-1}\big] \equiv 0, \quad 1 \leqslant r \leqslant \frac{p+q-1}{2}. \tag{18}$$

To prove (17) we may assume that $p+q+2r-2<n$ and apply (9) with $k'=n-p-q-2r+2$ and $m'=p+q-2r+2$, and to prove (18) we may assume that $p+q+2r-1<n$ and apply (8) with $k'=n-p-q-2r+1$ and $m'=p+q-2r+1$.

Subcase 2b. Assuming that $p+q$ is even, (16) means that

$$\big[a^{n-k}, a \circ c_{m-2}\big] \in \big\langle \big[a^{p+q}, a \circ c_{p+q-2}\big], \big[a^{p+q+1}, c_{p+q-2}\big], \big[a^{p+q+2}, a \circ c_{p+q-4}\big], \\ \ldots, \big[a^{2p+2q-2}, a \circ c_0\big], \big[a^{2p+2q-1}, c_0\big]\big\rangle$$

and so we have to prove that

$$\Delta_{m-1}b_{2(n-p-q)-1}\big[a^{p+q+2r-2}, a \circ c_{p+q-2r}\big] \equiv 0, \quad 1 \leqslant r \leqslant \frac{p+q}{2}, \tag{19}$$

and

$$\Delta_{m-1}b_{2(n-p-q)-1}\big[a^{p+q+2r-1}, c_{p+q-2r}\big] \equiv 0, \quad 1 \leqslant r \leqslant \frac{p+q}{2}. \tag{20}$$

To prove (19) we may assume that $p+q+2r-2<n$ and apply (8) with $k'=n-p-q-2r+2$ and $m'=p+q-2r+2$, and to prove (20) we may assume that $p+q+2r-1<n$ and apply (9) with $k'=n-p-q-2r+1$ and $m'=p+q-2r+1$.

The proof of (10) is thereby complete. If $m=n$ then (10) already gives the desired conclusion, so we may assume that $m<n$. Consider (7) with $l=n-m-1$ (since $m<n$ and m is even we indeed have $0\leqslant l\leqslant n-3$). Multiply this relation by Δ_{m-1}. Note that, in view of (10), the relation so obtained reduces to

$$\sum_{k=1}^{n-m}\Delta_{m-1}\binom{n+s}{k+s}b_{k+n-m-1}\bigl[a^{n-k},a\circ c_{m-2}\bigr]\equiv 0,\quad 0\leqslant s\leqslant n-m-1.$$

One can express these relations through the matrix A_{n-m}. Accordingly, arguing similarly as in the case when $m=1$, it follows that

$$\Delta_{m-1}\delta_{n-m}b_{k+n-m-1}\bigl[a^{n-k},a\circ c_{m-2}\bigr]\equiv 0,\quad 1\leqslant k\leqslant n-m.$$

This, together with (10), proves (i).

(ii) The proof for the case when m is odd is similar and so we give only its very brief outline. The crucial step of the proof is to show that

$$\Delta_{m-1}b_{k+n-m-1}\bigl[a^{n-k},c_{m-1}\bigr]\equiv 0,\quad n-m+1\leqslant k\leqslant n-1. \tag{21}$$

Now we have $m=2p+1$, $p\geqslant 1$, and again we consider two different cases, namely when $n-k=2q-1$ is odd and when $n-k=2q$ is even (we remark that the case when $k=n-1$ in (ii) does not require a different argument). In each case we have $1\leqslant q\leqslant p$. Set $t=p+q-1$ and use Lemma 2.4 in the first case, and Lemma 2.5 in the second case. In each case we consider separately the subcases when $p+q$ is odd or even, and of course we apply (8) and (9).

After proving (21) we may assume that $m<n$ and then consider (6) with $l=n-m-1$. The rest of the argument is essentially the same as in (i).

As already showed above, the last assertion in the formulation of the theorem is just another way of stating (i) and (ii) for $k=n-1$. □

3. Applications

Before treating some concrete question we shall first state two immediate corollaries to Theorem 2.1, the first one concerning derivations and the second one concerning generalized derivations. The first result is in fact just a special case of the second one, but certainly the most interesting case so we state it explicitly.

We partially keep the notation introduced above. In particular, n will be a fixed integer $\geqslant 2$ and Δ_m the corresponding integers. Unlike in the introduction where for simplicity we have considered algebras over fields, we continue to treat the case when $\mathcal{A}$ is an arbitrary ring. The definitions and observations concerning L_x, R_y and derivations, however, are the same in the ring-theoretic setting (just that of course we assume only the additivity of derivations instead of the linearity). By $\mathcal{E}(\mathcal{A})$ we denote the ring of all additive maps from a ring $\mathcal{A}$ into itself. Let $D \in \mathcal{E}(\mathcal{A})$ be a derivation. Pick any $x, y \in \mathcal{A}$ and let $\mathcal{G}$ be the additive subgroup of $\mathcal{E}(\mathcal{A})$ generated by all elements of the form $D^{n+k}L_{D^i(x)}R_{D^j(y)}$, $L_{D^i(x)}D^{n+k}R_{D^j(y)}$, $R_{D^j(y)}D^{n+k}L_{D^i(x)}$ and $L_{D^i(x)}R_{D^j(y)}D^{n+k}$ where i, j, k are nonnegative integers with $i + j + k = n - 2$.

From the last part of Theorem 2.1 we immediately infer

Corollary 3.1. *Let $m \leqslant n$ be a positive integer.*

(i) *If m is even then $\Delta_m L_{D^{2n-m-2}(x)}(D \circ R_{D^{m-1}(y)}) \in \mathcal{G}$.*
(ii) *If m is odd then $\Delta_m L_{D^{2n-m-2}(x)}R_{D^m(y)} \in \mathcal{G}$.*

A careful inspection of the proof of Theorem 2.1 allows one to obtain more explicit formulae. For example, when $n = 2$ and $m = 1$ one gets

$$2L_{D(x)}R_{D(y)} = D^2L_xR_y - L_xD^2R_y - R_yD^2L_x + L_xR_yD^2, \tag{22}$$

which can be easily verified and is in fact already known (see, e.g., [4, Observation 1]). For larger n's these formulae seem to be less tractable. For example, for $n = 4$ and $m = 1$ we have

$$\begin{aligned}2L_{D^5(x)}R_{D(y)} = {} & 2D^6L_xR_y - 6D^5L_{D(x)}R_y + 5D^4L_{D^2(x)}R_y \\ & - 2L_xD^6R_y - 6L_{D(x)}D^5R_y - 5L_{D^2(x)}D^4R_y \\ & - 2R_yD^6L_x + 6R_yD^5L_{D(x)} - 5R_yD^4L_{D^2(x)} \\ & + 2L_xR_yD^6 + 6L_{D(x)}R_yD^5 + 5L_{D^2(x)}R_yD^4,\end{aligned}$$

and for $n = 4$ and $m = 3$ we have

$$\begin{aligned}&10L_{D^3(x)}R_{D^3(y)} \\ &\quad = 2D^6L_xR_y - 6D^5L_{D(x)}R_y - 6D^5L_xR_{D(y)} + 15D^4L_{D(x)}R_{D(y)} \\ &\qquad - 2L_xD^6R_y - 6L_{D(x)}D^5R_y + 6L_xD^5R_{D(y)} + 15L_{D(x)}D^4R_{D(y)} \\ &\qquad - 2R_yD^6L_x + 6R_yD^5L_{D(x)} - 6R_{D(y)}D^5L_x + 15R_{D(y)}D^4L_{D(x)} \\ &\qquad + 2L_xR_yD^6 + 6L_{D(x)}R_yD^5 + 6L_xR_{D(y)}D^5 + 15L_{D(x)}R_{D(y)}D^4.\end{aligned} \tag{23}$$

We have considered only associative rings and algebras so far. Let us mention that the assumption that $\mathcal{A}$ in Corollary 3.1 is associative can be replaced by a slightly milder assumption that $(D^i(x)r)D^j(y) = D^i(x)(rD^j(y))$ for all $r \in \mathcal{A}$ and various choices of i and j (in order to guarantee that $L_{D^i(x)}$ and $R_{D^j(y)}$ commute).

By a *generalized derivation* on a ring $\mathcal{A}$ we shall mean a map $G \in \mathcal{E}(\mathcal{A})$ such that there exist a derivation D of $\mathcal{A}$ and $c \in \mathcal{A}$ such that $G = D + L_c$. The usual definition, as introduced in [4] and then used for instance in [9,12,13], is somewhat more general, but to make the paper less technical we shall use this simplified definition in the present paper; besides, in unital rings these two definitions coincide. A generalized derivation G is said to be inner if there are $a, b \in \mathcal{A}$ such that $G = L_a - R_b$. Inner generalized derivations have been studied mostly in operator theory.

We have to introduce some more notation. Given a generalized derivation $G = D + L_c$, we define another derivation $H = D + L_c - R_c$. Further, fix $x, y \in \mathcal{A}$ and denote by $\mathcal{G}$ the additive subgroup of $\mathcal{E}(\mathcal{A})$ generated by all elements of the form $G^{n+k} L_{H^i(x)} R_{D^j(y)}$, $L_{H^i(x)} G^{n+k} R_{D^j(y)}$, $R_{D^j(y)} G^{n+k} L_{H^i(x)}$ and $L_{H^i(x)} R_{D^j(y)} G^{n+k}$ where i, j, k are nonnegative integers with $i + j + k = n - 2$. Note that $L_{H^i(x)} = [G, L_{H^{i-1}(x)}]$ and $R_{D^i(y)} = [G, R_{D^{i-1}(y)}]$ for every $i \geqslant 1$. Therefore, letting G, L_x and R_y playing the roles of a, b_0 and c_0, Theorem 2.1 yields the following corollary.

Corollary 3.2. *Let $m \leqslant n$ be a positive integer.*

(i) *If m is even then $\Delta_m L_{H^{2n-m-2}(x)} (G \circ R_{D^{m-1}(y)}) \in \mathcal{G}$.*
(ii) *If m is odd then $\Delta_m L_{H^{2n-m-2}(x)} R_{D^m(y)} \in \mathcal{G}$.*

Replacing the roles of L_x and R_y at the beginning (i.e., letting L_x to play the role of c_0 and R_y the role of b_0) we obtain analogous statements, in particular $\Delta_m L_{H^m(x)} R_{D^{2n-m-2}(y)} \in \mathcal{G}$ for every odd $m \leqslant n$.

3.1. Nilpotent and related (generalized) derivations

One can easily construct nilpotent derivations. Let $a \in \mathcal{A}$ be such that $a^m = 0$. Using (1) we see that the inner derivation $D = L_a - R_a$ then satisfies $D^{2m-1} = 0$. There are many results in the literature (see, e.g., [8,15,16]) showing that every nilpotent derivation essentially arises in this way. In order to state one of them we have to introduce some more notation and terminology. Let $\mathcal{A}$ be a prime ring, i.e., a ring in which $L_a R_b = 0$, where $a, b \in \mathcal{A}$, implies $a = 0$ or $b = 0$. By $\mathcal{Q}_s(\mathcal{A})$ we shall denote the symmetric Martindale ring of quotients of $\mathcal{A}$, and by C the extended centroid of $\mathcal{A}$. For definition and basic properties of these and some related concepts that will be used we refer the reader to [2]. Let us just mention that $\mathcal{Q}_s(\mathcal{A})$ is a prime ring containing $\mathcal{A}$ as its subring, C is its center and it is a field. Recall that a derivation D is said to be X-inner if $D = L_a - R_a$ for some $a \in \mathcal{Q}_s(\mathcal{A})$ (we shall use the notation L_a and R_a even when a can lie in a ring bigger than $\mathcal{A}$).

Theorem 3.3. *Let $\mathcal{A}$ be a prime ring, let D be a derivation of $\mathcal{A}$, let $n \geqslant 1$ and suppose that* $\mathrm{char}(\mathcal{A}) = 0$ *or* $\mathrm{char}(\mathcal{A}) > n$. *Then $D^n = 0$ if and only if there exists $a \in \mathcal{Q}_s(\mathcal{A})$ such that $D = L_a - R_a$ and $a^{[(n+1)/2]} = 0$.*

Theorem 3.3 follows from [11, Theorem 1]; its proof, however, rests heavily on results from [10] and [15]. Using Corollary 3.2 we are now in a position to extend this result to generalized derivations.

Theorem 3.4. *Let $\mathcal{A}$ be a prime ring and let G be a generalized derivation of $\mathcal{A}$. Let $n \geqslant 2$ and assume that* $\operatorname{char}(\mathcal{A}) = 0$ *or* $\operatorname{char}(\mathcal{A}) \geqslant 2n - 1$. *Then the following two conditions are equivalent*:

(i) $G^n = 0$.

(ii) *There exist $a, b \in \mathcal{Q}_s(\mathcal{A})$ and a nonnegative integer $s \leqslant n - 1$ such that $G = L_a - R_b$, $a^{s+1} = 0$, and $b^{n-s} = 0$.*

Proof. It is trivial to check that (ii) implies (i). Assume that (i) holds. We shall use the notation introduced before the statement of Corollary 3.2. If $D = 0$ then $G = L_c$ where $c \in \mathcal{A}$ and $c^n = 0$, and so (ii) holds true for $a = c$, $b = 0$ and $s = n - 1$. So we assume that $D \neq 0$, and a similar argument shows that we can also assume that $H \neq 0$. By Corollary 3.2 we have $\Delta_1 L_{H^{2n-3}(x)} R_{D(y)} = 0$ for all $x, y \in \mathcal{A}$, which implies that $H^{2n-3} = 0$ since $\mathcal{A}$ is prime. Therefore, by the result stated above, there is $a \in \mathcal{Q}_s(\mathcal{A})$ such that $H = L_a - R_a$ and $a^{n-1} = 0$. Let s be an integer such that $a^{s+1} = 0$ and $a^s \neq 0$. Of course $s \leqslant n - 2$, and also $s \geqslant 1$ since $H \neq 0$. Note that $H^{2s-1} L_a = (-1)^s \binom{2s-1}{s} L_{a^s} R_{a^s} \neq 0$ and so $H^{2s-1} \neq 0$. By Corollary 3.2 (and the remark following its statement) it follows that $\Delta_r L_{H^{2s-1}(x)} R_{D^{2n-2s-1}(y)} = 0$ for all $x, y \in \mathcal{A}$ where $r = 2n - 2s - 1$ (if $2n - 2s - 1 \leqslant n$) or $r = 2s - 1$ (if $2n - 2s - 1 > n$). Hence it follows that $D^{2n-2s-1} = 0$ and so Theorem 3.3 tells us that there exists $b \in \mathcal{Q}_s(\mathcal{A})$ such that $D = L_b - R_b$ and $b^{n-s} = 0$. Since $L_a - R_a = H = D + L_c - R_c = L_{c+b} - R_{c+b}$ we have $[c + b - a, \mathcal{A}] = 0$ from which it follows that $\lambda = c + b - a \in C$ (see, e.g., [2, Remark 2.3.1]). Therefore $G = L_c + D = L_{c+b} - R_b = \lambda I + L_a - R_b$. In view of $a^{s+1} = 0$ and $b^{n-s} = 0$ we have $(L_a - R_b)^n = 0$. Since $G^n = 0$ as well it follows that $\lambda^{2n-1} I = (\lambda I)^{2n-1} = (G - (L_a - R_b))^{2n-1} = 0$. However, C is a field and so $\lambda = 0$ which completes the proof of (ii). □

If n in Theorem 3.3 is an even number, then it follows immediately from the conclusion of this theorem that $D^{n-1} = 0$. Chung and Luh [6] proved this under a milder condition: If D is a derivation of a 2-torsionfree semiprime ring, then $D^{2k} = 0$ implies $D^{2k-1} = 0$ (recall that a ring $\mathcal{A}$ is m-torsionfree if $ma = 0$ with $a \in \mathcal{A}$ implies $a = 0$). A related, but analytic result was obtained by Turovskii and Shulman [18]. They proved that if D is a derivation on a Banach algebra $\mathcal{A}$, then $D^{2k}(\mathcal{A}) \subseteq J(\mathcal{A})$, where $J(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$, implies $D^{2k-1}(\mathcal{A}) \subseteq J(\mathcal{A})$. Recently Beidar and Brešar [1] generalized this result to arbitrary algebras over fields with some restrictions concerning their characteristic.

The proofs of these related results are quite different. Corollary 3.1 makes it possible to unify and generalize them. Setting $n = 2k$ and $m = 2k - 1$ we immediately get

Theorem 3.5. *Let $\mathcal{A}$ be a ring and let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Suppose that D is a derivation of $\mathcal{A}$ such that D^n maps $\mathcal{A}$ into $\mathcal{I}$. If $n = 2k$ is even, then $\Delta_{2k-1} L_{D^{2k-1}(x)} R_{D^{2k-1}(y)}$ maps $\mathcal{A}$ into $\mathcal{I}$ for any $x, y \in \mathcal{A}$. In particular, if $\mathcal{I}$ is a semiprime ideal and the ring $\mathcal{A}/\mathcal{I}$ is $(4k - 2)!$-torsionfree, then D^{2k-1} maps $\mathcal{A}$ into $\mathcal{I}$.*

Letting $\mathcal{I} = 0$ we thus have: if $D^{2k} = 0$ then $\Delta_{2k-1} D^{2k-1}(\mathcal{A}) \mathcal{A} D^{2k-1}(\mathcal{A}) = 0$. If $\mathcal{A}$ is not semiprime then it does not necessarily follow that $D^{2k-1} = 0$ (even when $\operatorname{char}(\mathcal{A}) = 0$).

For example, if $\mathcal{A}$ is the ring of all $2k \times 2k$ upper triangular matrices over a field F and $a = e_{12} + e_{23} + \cdots + e_{2k-1,2k} \in \mathcal{A}$ (where e_{ij} denotes the matrix unit), then the derivation $D = L_a - R_a$ satisfies $D^{2k} = 0$, $D^{2k-1} \neq 0$, and $D^{2k-1}(\mathcal{A})\mathcal{A}D^{2k-1}(\mathcal{A}) = 0$.

Incidentally we remark that the above arguments clearly show that, as already mentioned in the introduction, Theorem 1.1 does not hold true in the case when m_1 and m_2 are even. Namely, if it was true then under rather mild assumptions we could show that $D^{2k+1} = 0$ always implies $D^{2k} = 0$ which is of course false.

Let $\mathcal{I}$ be a nonzero ideal of a ring $\mathcal{A}$, and suppose that $D^n(\mathcal{I}) = 0$. Chung and Luh [7] proved that then $D^n(\mathcal{A}) = 0$ if $\mathcal{A}$ is a prime ring. Our results enable to consider this condition in arbitrary rings. Indeed, from Corollary 3.1 it follows that for all $x, y \in \mathcal{A}$ we have $\Delta_n L_{D^{n-2}(x)} R_{D^n(y)}(\mathcal{I}) = 0$ if n is odd, and $\Delta_{n-1} L_{D^{n-1}(x)} R_{D^{n-1}(y)}(\mathcal{I}) = 0$ if n is even. Since $\Delta_{n-1} = \Delta_n$ in any case we can say the following.

Theorem 3.6. *Let $\mathcal{A}$ be a ring and let $\mathcal{I}$ be an ideal of $\mathcal{A}$. If D is a derivation of $\mathcal{A}$ such that $D^n(\mathcal{I}) = 0$, then $\Delta_n D^n(\mathcal{A})\mathcal{I}D^n(\mathcal{A}) = 0$.*

If we assume in addition that $\mathcal{A}$ is $(2n-2)!$-torsionfree semiprime ring and $\mathcal{I}$ is its essential ideal, then it follows that $D^n(\mathcal{A}) = 0$.

3.2. Powers of (generalized) derivations having finite rank

In 1983, Bergen [3] proved that if a derivation D of an infinite dimensional prime algebra is such that D^n has finite rank, then $D^{2n-1} = 0$. In the recent paper [5] the first two authors obtained a considerably more detailed conclusion (see Theorem 3.8 below). Our first goal is to improve this result. More precisely, we shall obtain a similar but more intrinsic conclusion, and simultaneously extend it to generalized derivations (Theorem 3.10).

We need some definitions and results from [5] and [14]. Let $\mathcal{A}$ be a semiprime algebra over a field F. It is well known that a left ideal $\mathcal{L}$ of $\mathcal{A}$ is minimal if and only if there exists an idempotent $e \in \mathcal{A}$ such that $\mathcal{L} = \mathcal{A}e$ and $e\mathcal{A}e$ is a division algebra. The socle of $\mathcal{A}$ is defined as the sum of all minimal left ideals of $\mathcal{A}$. Further, the *lower socle* of $\mathcal{A}$ is defined as the sum of all minimal left ideals $\mathcal{A}e$ of $\mathcal{A}$ such that $\dim_F(e\mathcal{A}e) < \infty$ [5]. We shall denote the socle of $\mathcal{A}$ by $\operatorname{soc}(\mathcal{A})$ and the lower socle of $\mathcal{A}$ by $\underline{\operatorname{soc}}_F(\mathcal{A})$. Using the semiprimeness of $\mathcal{A}$ it is easy to see that both $\operatorname{soc}(\mathcal{A})$ and $\underline{\operatorname{soc}}_F(\mathcal{A})$ are ideals (in fact, both definitions are left-right symmetric). In general we of course have $\underline{\operatorname{soc}}_F(\mathcal{A}) \subseteq \operatorname{soc}(\mathcal{A})$. In the case $\mathcal{A}$ is prime it is easy to establish (see [5]) that $\underline{\operatorname{soc}}_F(\mathcal{A})$ can be different from $\operatorname{soc}(\mathcal{A})$ only in the case when $\underline{\operatorname{soc}}_F(\mathcal{A}) = 0$. Let us now state [5, Theorems 3.3 and 3.4]; we remark that almost identical results were also obtained independently in a remarkably similar paper [14] (though in [14] the concept of the lower socle is not explicitly defined).

Lemma 3.7 [5,14]. *If $\mathcal{A}$ is semiprime, then $a \in \underline{\operatorname{soc}}_F(\mathcal{A})$ if and only if the operator $L_a R_a$ has finite rank. Moreover, if $\mathcal{A}$ is prime and $a, b \in \mathcal{A}$ are nonzero, then $a, b \in \underline{\operatorname{soc}}_F(\mathcal{A})$ if and only if $L_a R_b$ has finite rank.*

Assume now that $\mathcal{A}$ is a prime algebra. The base field F can be viewed as a subfield of the extended centroid C, and the central closure $\mathcal{A}C$ (i.e., the subring of $\mathcal{Q}_s(\mathcal{A})$ generated by $\mathcal{A}$ and C) can be regarded as an algebra over C as well as an algebra over F. We can now state

Theorem 3.8 [5]. *Let $\mathcal{A}$ be an infinite dimensional prime algebra over a field F, let D be a derivation of $\mathcal{A}$, and let $n \geqslant 1$ be such that* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F) \geqslant 2n$. *If D^n has finite rank and $D^n \neq 0$, then $n \geqslant 2$ and there exist $a \in \mathcal{Q}_s(\mathcal{A})$ and an integer s such that $D = L_a - R_a$, $(n-1)/2 < s < n$, $a^{s+1} = 0$, $a^s \neq 0$, and $a^{n-s} \in \underline{\mathrm{soc}}_C(\mathcal{A}C)$.*

Theorem 3.8 follows from a slightly more general result [5, Theorem 5.3] which deals with the assumption that $D^n(\mathcal{A})$ is contained in a finite dimensional subspace of $\mathcal{Q}_s(\mathcal{A})$ where $\mathcal{Q}_s(\mathcal{A})$ is considered as an algebra over C. This explains why the extended centroid and the central closure appear in the conclusion of Theorem 3.8. It seems natural to ask whether their presence can be avoided in the case when the assumptions deal only with the base field F (as in Theorem 3.8), more precisely, whether the role of $\underline{\mathrm{soc}}_C(\mathcal{A}C)$ can be replaced by $\underline{\mathrm{soc}}_F(\mathcal{A})$. We shall prove this is true indeed (Corollary 3.11), and so in particular $\mathcal{A}$ itself (not only its central closure) is a primitive ring with nonzero socle.

First we give some general comments on the connection between $\underline{\mathrm{soc}}_F(\mathcal{A})$ and $\underline{\mathrm{soc}}_C(\mathcal{A}C)$. It may happen that $\underline{\mathrm{soc}}_F(\mathcal{A}) = 0$ while $\underline{\mathrm{soc}}_C(\mathcal{A}C) \neq 0$. For example, if $\mathcal{A} = M_n(E)$ where the field E is an infinite extension of the field F, then $C = E$ and $\underline{\mathrm{soc}}_C(\mathcal{A}C) = \mathcal{A}C = \mathcal{A}$ while $\underline{\mathrm{soc}}_F(\mathcal{A}) = 0$; moreover, if we consider for example $\mathcal{A} = M_n(F[X])$ then $\mathrm{soc}(\mathcal{A}) = 0$ (and so of course also $\underline{\mathrm{soc}}_F(\mathcal{A}) = 0$) while $\underline{\mathrm{soc}}_C(\mathcal{A}C) = \mathcal{A}C = M_n(F(X))$ is even bigger than $\mathcal{A}$. On the other hand, if $\underline{\mathrm{soc}}_F(\mathcal{A}) \neq 0$ then $\underline{\mathrm{soc}}_F(\mathcal{A}) = \underline{\mathrm{soc}}_C(\mathcal{A}C)$. Indeed, in that case we have $\underline{\mathrm{soc}}_F(\mathcal{A}) = \mathrm{soc}(\mathcal{A})$ and so $\mathcal{A}$ is a primitive algebra which yields that $\mathrm{soc}(\mathcal{A}) = \mathrm{soc}(\mathcal{A}C)$ by [2, Theorem 4.3.6(ii)]. Now given any minimal idempotent $e \in \mathrm{soc}(\mathcal{A}C)$ $(= \mathrm{soc}(\mathcal{A}))$ we have that $\dim_F(e\mathcal{A}e) < \infty$ from which we infer that $\dim_C(e\mathcal{A}Ce) < \infty$, thus $\underline{\mathrm{soc}}_C(\mathcal{A}C) \neq 0$ and so $\underline{\mathrm{soc}}_C(\mathcal{A}C) = \mathrm{soc}(\mathcal{A}C)$. Accordingly, $\underline{\mathrm{soc}}_F(\mathcal{A}) = \mathrm{soc}(\mathcal{A}) = \underline{\mathrm{soc}}_C(\mathcal{A}C)$.

We shall also need the following result, a somewhat simplified version of [5, Corollary 4.4] (a very similar result, though not sufficient for our purposes, was obtained also in [14]).

Lemma 3.9 [5]. *Let $\mathcal{A}$ be a prime algebra. Suppose that $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ are C-independent subsets of $\mathcal{Q}_s(\mathcal{A})$ such that $\sum_{i=1}^r L_{a_i} R_{b_i}$ maps $\mathcal{A}$ into some finite dimensional C-subspace of $\mathcal{Q}_s(\mathcal{A})$. Then $a_i, b_i \in \underline{\mathrm{soc}}_C(\mathcal{A}C)$ for all $i = 1, \dots, r$.*

Theorem 3.10. *Let $\mathcal{A}$ be an infinite dimensional prime algebra over a field F, let G be a generalized derivation of $\mathcal{A}$, and let $n \geqslant 1$ be such that* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F) \geqslant 2n$. *Then the following two conditions are equivalent*:

(i) *G^n has finite rank and $G^n \neq 0$;*
(ii) *$n \geqslant 2$ and there exist $a, b \in \mathcal{Q}_s(\mathcal{A})$ and integers s, t such that $G = L_a - R_b$, $0 < s, t < n$, $s + t \geqslant n$, $a^{s+1} = 0$, $a^s \neq 0$, $a^{n-t} \in \underline{\mathrm{soc}}_F(\mathcal{A})$, $b^{t+1} = 0$, $b^t \neq 0$, and $b^{n-s} \in \underline{\mathrm{soc}}_F(\mathcal{A})$.*

Proof. Assume that (i) holds. As above we set $G = D + L_c$ and $H = D + L_c - R_c$. Given any $y \in \mathcal{A}$ we have $R_{D^{2n-1}(y)} = [G, R_{D^{2n-2}(y)}] = [G, [G, [G, \ldots, [G, R_y] \ldots]]]$, that is,

$$R_{D^{2n-1}(y)} = \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} G^{2n-1-i} R_y G^i.$$

Since G^n has finite rank it follows that $R_{D^{2n-1}(y)}$ has finite rank too, i.e., the left ideal $\mathcal{A}D^{2n-1}(y)$ is finite dimensional. However, an infinite dimensional prime algebra cannot contain nonzero finite dimensional left ideals [14, Theorem 1.7] (the proof is simple: if $\mathcal{I}$ was a nonzero finite dimensional left ideal of $\mathcal{A}$, then $a \mapsto L_a$ would be an embedding of $\mathcal{A}$ into the finite dimensional algebra $\mathcal{L}(\mathcal{I})$ of linear operators on $\mathcal{I}$). Therefore $D^{2n-1} = 0$. The same argument shows that $H^{2n-1} = 0$. By Theorem 3.3 there exist $a, b \in \mathcal{Q}_s(\mathcal{A})$ such that $H = L_a - R_a$, $D = L_b - R_b$, $a^n = 0$, and $b^n = 0$. Let $0 \leqslant s, t < n$ be such that $a^{s+1} = 0$, $a^s \neq 0$, $b^{t+1} = 0$, $b^t \neq 0$. As in the proof of Theorem 3.4 we see that $\lambda = c + b - a \in C$ and $G = L_{a+\lambda} - R_b$. Suppose that $\lambda \neq 0$. Then $\lambda + a$ is invertible in $\mathcal{Q}_s(\mathcal{A})$ since a is nilpotent. Let $u \in \mathcal{A}$ be such that $0 \neq ub^t \in \mathcal{A}$ (see [2, Proposition 2.2.3]). Note that $L_{(\lambda+a)^{-n}} G^n R_{ub^t} = R_{ub^t}$. Since G^n has finite rank it follows that R_{ub^t} has finite rank too, a contradiction. Therefore $\lambda = 0$ and so $G = L_a - R_b$. In particular, it is now clear that n cannot be 1, and that $s, t \neq 0$ and $s + t \geqslant n$ since $G^n \neq 0$. Note that $G^n = \sum_{i=n-s}^{t} (-1)^i \binom{n}{i} L_{a^{n-i}} R_{b^i}$. Since $\dim_F G^n(\mathcal{A}) < \infty$ and F is a subfield of C, $\sum_{i=n-s}^{t} (-1)^i \binom{n}{i} L_{a^{n-i}} R_{b^i}$ of course maps $\mathcal{A}$ into a finite dimensional subspace of $\mathcal{A}C$. The sets $\{a^{n-t}, a^{n-t+1}, \ldots, a^s\}$ and $\{b^{n-s}, b^{n-s+1}, \ldots, b^t\}$ are C-independent since a and b are nilpotent, and so it follows from Lemma 3.9 that all these elements lie in $\underline{\mathrm{soc}}_C(\mathcal{A}C)$. In particular, $a^{n-t}, b^{n-s} \in \underline{\mathrm{soc}}_C(\mathcal{A}C)$. It remains to prove that they actually belong to $\underline{\mathrm{soc}}_F(\mathcal{A})$. In view of the above remarks it suffices to show that $\underline{\mathrm{soc}}_F(\mathcal{A}) \neq 0$. Since $a^s \neq 0$ and $b^t \neq 0$ one can easily show (as in the proof of Theorem 3.4) that $H^{2s-1} \neq 0$ and $D^{2t-1} \neq 0$. Therefore also $D^{2n-2s-1} \neq 0$ since $s + t \geqslant n$. Similarly as in Theorem 3.4 we now apply Corollary 3.2 to obtain that $\Delta_r L_{H^{2s-1}(x)} R_{D^{2n-2s-1}(y)}$ has finite rank for all $x, y \in \mathcal{A}$ where $r = 2n - 2s - 1$ (if $2n - 2s - 1 \leqslant n$) or $r = 2s - 1$ (if $2n - 2s - 1 > n$). By Lemma 3.7 it follows that $H^{2s-1}(x), D^{2n-2s-1}(y) \in \underline{\mathrm{soc}}_F(\mathcal{A})$ for all $x, y \in \mathcal{A}$. In particular this proves that $\underline{\mathrm{soc}}_F(\mathcal{A}) \neq 0$, as desired.

Conversely, assume that (ii) holds. Then $G^n = \sum_{i=n-s}^{t} (-1)^i \binom{n}{i} L_{a^{n-i}} R_{b^i}$. Since $\underline{\mathrm{soc}}_F(\mathcal{A})$ is an ideal of $\mathcal{A}$ containing the elements a^{n-t} and b^{n-s}, it contains all elements a^{n-i}, b^i, $i = n-s, \ldots, t$. Therefore Lemma 3.7 tells us that $L_{a^{n-i}} R_{b^i}$ has finite rank for each $i = n-s, \ldots, t$. But then G^n has finite rank too. □

Corollary 3.11. *Let $\mathcal{A}$ be an infinite dimensional prime algebra over a field F, let D be a derivation of $\mathcal{A}$, and let $n \geqslant 1$ be such that* $\operatorname{char}(F) = 0$ *or* $\operatorname{char}(F) \geqslant 2n$. *Then the following two conditions are equivalent*:

- (i) *D^n has finite rank and $D^n \neq 0$;*
- (ii) *$n \geqslant 2$ and there exist $a \in \mathcal{Q}_s(\mathcal{A})$ and an integer s such that $D = L_a - R_a$, $(n-1)/2 < s < n$, $a^{s+1} = 0$, $a^s \neq 0$, and $a^{n-s} \in \underline{\mathrm{soc}}_F(\mathcal{A})$.*

In particular, in this case $\underline{\mathrm{soc}}_F(\mathcal{A}) \neq 0$, $D^{2s+1} = 0$ and $D^{2n-2s-1}$ maps into $\underline{\mathrm{soc}}_F(\mathcal{A})$.

Let us show by an example that the last assertion, namely $D^{2s+1} = 0$ and $D^{2n-2s-1}(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$ where $(n-1)/2 < s < n$, is not equivalent to (i) and (ii).

Example 3.12. Let $\mathcal{V}$ be an infinite dimensional vector space over a field F with $\mathrm{char}(F) \neq 2$, and let $b \in \mathcal{L}(\mathcal{V})$ be any operator of infinite rank such that $b^2 = 0$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{L}(\mathcal{V})$ generated by b and all finite rank linear operators. Then the inner derivation $D = L_b - R_b$ satisfies $D^3 = 0$, $D^2 \neq 0$, and $D(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$; however, the rank of D^2 is not finite.

In general we cannot claim that in Corollary 3.11 the element a lies in $\mathcal{A}$, so the presence of some rings of quotients is really necessary:

Example 3.13. Let $\mathcal{V}$ and b be as in Example 3.12, while by $\mathcal{A}$ we denote the algebra of all finite rank linear operators on $\mathcal{V}$. Moreover, let $c \in \mathcal{A}$ be such that $bc = cb = 0$, $c^2 \neq 0$ and $c^3 = 0$ (concrete examples of such b and c can be easily found), and set $a = b + c$. Note that $D = L_a - R_a$ is a derivation from $\mathcal{A}$ into itself (which is not inner but just X-inner), and that $a \notin \mathcal{A}$, $0 \neq a^2 = c^2 \in \underline{\mathrm{soc}}_F(\mathcal{A}) = \mathcal{A}$ and $a^3 = 0$. Therefore $D^4 \neq 0$ and has finite rank.

Let $\mathcal{A}$ be as in Corollary 3.11 and let D^n have finite rank. Then it follows that $D^{n-1}(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$, unless of course $D^n = 0$. In any case we have $D^n(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$, and moreover $D^{n-1}(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$ if n is even [6]. This observation can be extended to semiprime algebras.

Theorem 3.14. *Let $\mathcal{A}$ be a semiprime algebra over a field F, let D be a derivation of $\mathcal{A}$, and let $n \geqslant 1$ be such that* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F) \geqslant 2n - 1$. *If D^n has finite rank then $D^n(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$. Moreover, if n is even then $D^{n-1}(\mathcal{A}) \subseteq \underline{\mathrm{soc}}_F(\mathcal{A})$.*

Proof. Let n be odd. If $n > 1$ then by Corollary 3.1, $\Delta_n L_{D^{n-2}(x)} R_{D^n(y)}$ has finite rank for any $x, y \in \mathcal{A}$, and so in particular $L_{D^n(y)} R_{D^n(y)}$ has finite rank. The same is true for $n = 1$ since in this case both $L_{D(y)} = [D, L_y]$ and $R_{D(y)} = [D, R_y]$ have finite rank. Therefore $D^n(y) \in \underline{\mathrm{soc}}_F(\mathcal{A})$ by Lemma 3.7. If n is even, then Corollary 3.1 implies that $\Delta_{n-1} L_{D^{n-1}(y)} R_{D^{n-1}(y)}$ has always finite rank, and hence $D^{n-1}(y) \in \underline{\mathrm{soc}}_F(\mathcal{A})$. □

Theorem 3.14 settles the problem mentioned in [5]. The special case when $n = 2$ was already handled in [5, Corollary 5.6].

3.3. Derivations on Banach algebras

A Banach algebra $\mathcal{A}$ is said to be *ultrasemiprime* if there exists a constant $\kappa_{\mathcal{A}} > 0$ such that $\|L_a R_a\| \geqslant \kappa_{\mathcal{A}} \|a\|^2$ for every $a \in \mathcal{A}$. For example, if $\mathcal{A}$ is a C*-algebra then $\|aa^*a\| = \|a\|^3$ for every $a \in \mathcal{A}$, which shows that $\mathcal{A}$ is ultrasemiprime (with $\kappa_{\mathcal{A}} = 1$).

Let D be a derivation of an ultrasemiprime Banach algebra $\mathcal{A}$. In [4, Theorem 2, Remark 4] it was proved that $\|D^2\| \geqslant \kappa_{\mathcal{A}}\frac{1}{2}\|D\|^2$. This follows easily from (22). Indeed, using this identity we get

$$2\kappa_{\mathcal{A}}\bigl\|D(x)\bigr\|^2 \leqslant \|2L_{D(x)}R_{D(x)}\| \leqslant 4\|L_x\|\|R_x\|\bigl\|D^2\bigr\| \leqslant 4\|x\|^2\bigl\|D^2\bigr\|,$$

which implies this inequality. We can now extend this to higher powers of derivations. For example, using (23) a similar argument shows that $\|D^4\| \geqslant \kappa_{\mathcal{A}}(10/116)(\|D^3\|/\|D\|)^2$. So, arguing in this manner we see that there is a sequence of positive real numbers (c_k) (independent of $\mathcal{A}$) such that

$$\bigl\|D^{2k}\bigr\| \geqslant \kappa_{\mathcal{A}}c_k\left(\frac{\|D^{2k-1}\|}{\|D\|^{k-1}}\right)^2$$

for each positive integer k. This result can be viewed as an analytic version of the result of Chung and Luh [6].

Finally we just indicate another area where our results are applicable. Recall that an element a from a Banach algebra $\mathcal{A}$ is said to be *compact* if L_aR_a is a compact operator on $\mathcal{A}$. We denote the set of all compact elements in $\mathcal{A}$ by $K(\mathcal{A})$. We remark that from $K(\mathcal{A}) \neq 0$ we can often get some useful information on the structure of the algebra (for example, if $\mathcal{A}$ is a C*-algebra then $K(\mathcal{A}) = \overline{\operatorname{soc}(\mathcal{A})}$ [17, Proposition 2.1] and so, in particular, $K(\mathcal{A}) \neq 0$ implies $\operatorname{soc}(\mathcal{A}) \neq 0$). Since compact operators from $\mathcal{A}$ into $\mathcal{A}$ form an ideal of the algebra of all bounded linear operators on $\mathcal{A}$, a simple modification of the proof of Theorem 3.14 yields the following result: if D^n is a compact operator on $\mathcal{A}$ then $D^n(\mathcal{A}) \subseteq K(\mathcal{A})$; moreover, if n is even then $D^{n-1}(\mathcal{A}) \subseteq K(\mathcal{A})$ (in particular, $K(\mathcal{A}) \neq 0$ provided that $D^n \neq 0$). This result will be used in the forthcoming paper of the first author and Yu. Turovskii, where the problem of determining derivations whose power is compact will be thoroughly studied.

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