Partial Galois theory of commutative rings
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Abstract

In this article we develop a Galois theory of commutative rings under partial actions of finite groups, extending the well-known results by S. U. Chase, D. K. Harrison and A. Rosenberg.

1. Introduction

In the celebrated paper by S. U. Chase, D. K. Harrison and A. Rosenberg [2] the authors developed a Galois theory for commutative ring extensions $S \supset R$, under the assumptions that S is separable over R, finitely generated and projective as an R-module, and the elements of the Galois group G are pairwise strongly distinct R-automorphisms of S. Among other results their Theorem 1.3 gives several equivalent conditions for the definition of a Galois extension and Theorem 2.3 states a one-to-one correspondence between the subgroups of G and the R-subalgebras of S which are separable and G-strong.

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On the other hand, partial actions of groups have been introduced in the theory of operator algebras giving powerful tools of their study (see, in particular, [8], [9], [12], [16] and [17]). A related concept, that of a partial representation of a group on a Hilbert space, have been defined independently by R. Exel [9], and J. C. Quigg and I. Raeburn [17]. Several relevant classes of C^*-algebras, were deeply investigated in [10], [11], [12] from the point of view of partial actions and partial representations of groups, including the Cuntz-Krieger algebras introduced in [3].

Partial actions on C^*-algebras (partial C^*-dynamical systems) grew out from the desire to extend C^*-crossed products, the most important steps being made in [7], [16] and [8]. The concept of a partial action of a group G on an abstract set X was introduced by R. Exel in [9] as a family of partial bijections of X satisfying natural compatibility conditions. It also can be defined as a partial homomorphism (partial representation) from G to the symmetric inverse semigroup $\mathcal{I}(X)$ of X (see [9]), which in this case is easily viewed as a premorphism [15], a rather natural concept studied in the theory of inverse semigroups. Recently, in a pure algebraic context, partial representations and partial actions of groups on algebras have been studied in [4], [5] and [6]. J. Kellendonk and M. Lawson [15] observed the relevance of partial actions for Fuchsian groups, model theory, group presentations, the Ribes-Zalesskii property of groups, $\mathbb{R}$-trees and various aspects in the theory of semigroups.

Given a partial action of a group on an object it is natural to ask whether it is a restriction of a global action defined on a bigger object. Such global action is called a globalization or an enveloping action, provided that a certain minimality condition is satisfied which guarantees its uniqueness. Globalizations of partial actions where first considered by F. Abadie in his PhD Thesis of 1999 (see also [1]). In the cases of abstract sets and general topological spaces enveloping actions always exist, however, when dealing with Hausdorf spaces a problem appears which implies that C^*-algebraic global extensions not always exist. Independently from [1] globalizations of partial actions were investigated in [15], including the cases of abstract sets, topological spaces and partially ordered sets. As a further development, B. Steinberg studied partial actions on cell complexes, viewed as combinatorial objects [18], establishing some interesting parallels with the Bass-Serre Theory of group actions on trees.

A partial action α of a group G on a unital algebra S is a collection of ideals S_σ, $\sigma \in G$, together with isomorphisms $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma, \sigma \in G$, which satisfy some additional conditions of compatibility with the group. From the categorical point of view it seems to be reasonable to suppose that the S_σ's and S are objects of the same category, i.e., each S_σ is a unital algebra. This idea is confirmed when dealing with globalizations: a partial action on a unital algebra possesses an enveloping action (which is necessarily unique) if and only if every S_σ is an algebra with unity element [5]. That this situation is natural in one more sense follows from the results of this article: assuming this condition a complete generalization of the results by Chase-Harrison-Rosenberg can be obtained in the context of partial actions.

The purpose of this paper is to develop a Galois theory for a commutative ring extension $S \supset R$, when G is a group acting partially on S by R-linear maps. It is almost unnecessary to say that a part of our results are proved using some ideas of [2], but it is required to check carefully many details which come from the fact that the action of the Galois group is partial instead of being global.

We shall deal with a partial action α of G on S which has an enveloping action, i.e., there exist a ring S' and a global action of G on S' such that S is an ideal of S' and the restriction of the global action to the ideals S_σ gives the partial action α [5].

Section 2 contains some preliminaries. In Section 3 we define partial Galois extensions and prove a theorem giving several equivalent conditions for S to be a partial Galois extension of the ring R, extending to our case Theorem 1.3 of [2]. In Section 4 we show that S is a partial Galois extension of R with group G and partial action α if and only if S' is a (global) Galois extension of S'^G with Galois group G. Section 5 is devoted to prove the Fundamental Galois Theorem for partial actions, giving an extension of Theorem 2.3 of [2]. Finally, in Section 6 we include some examples and additional remarks.

Throughout this paper R denotes a commutative ring with an identity element. Algebras are assumed to be commutative and associative with identity element and homomorphisms of unital algebras are always assumed to send the identity into the identity. Unadorned $\otimes$ means $\otimes_R$.

2. Prerequisites

We first recall the notion of a partial action of a group on an algebra [5].

Let G be a group and S a unital R-algebra. A *partial action* α of G on S is a collection of ideals S_σ, $\sigma \in G$, of S and isomorphisms of (non-necessarily unital) R-algebras $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma$ such that:

(i) $S_1 = S$ and α_1 is the identity automorphism of S,

(ii) $S_{(\sigma\tau)^{-1}} \supseteq \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$,

(iii) $\alpha_\sigma \circ \alpha_\tau(x) = \alpha_{\sigma\tau}(x)$, for every $x \in \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$ and $\sigma, \tau \in G$.

In the rest of the paper, to say that $\alpha = \{\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma \mid \sigma \in G\}$ is a partial action of a group G on an R-algebra S we will simply say that α is a partial action of G on S, where the ideals associated with the action will be denoted by S_σ, unless otherwise stated. We recall some facts which are already known (see [5]). The property (ii) of the definition easily implies that $\alpha_\sigma(S_{\sigma^{-1}} \cap S_\tau) = S_\sigma \cap S_{\sigma\tau}$, for all $\sigma, \tau \in G$. Also $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$, for every $\sigma \in G$.

From here on we assume that the group G is finite and any algebra S is a commutative faithful R-algebra with identity element. We identify R with $R1$ and so we may assume $R \subset S$. Furthermore, unless for some remarks in Section 6, each S_σ is supposed to be generated by an idempotent element 1_σ, i.e., S_σ is an R-algebra with identity. It is clear that in this case $S_\sigma \cap S_\tau = 1_\sigma 1_\tau S$. In the particular case that $S_\sigma = S$, for all $\sigma \in G$, we have a usual (global) action of the group G on the R-algebra S.

The following lemma will be very useful in the sequel.

Lemma 2.1 *Let α be a partial action of a group G on an R-algebra S such that each R-algebra S_σ has identity element 1_σ. Then the equalities*

(i) $\alpha_\sigma(1_{\sigma^{-1}} 1_\tau) = 1_\sigma 1_{\sigma\tau}$ *and*

(ii) $\alpha_\sigma(\alpha_\tau(x 1_{\tau^{-1}} 1_{\tau^{-1}\sigma^{-1}})) = \alpha_\sigma(\alpha_\tau(x 1_{\tau^{-1}}) 1_{\sigma^{-1}}) = \alpha_{\sigma\tau}(x 1_{\tau^{-1}\sigma^{-1}}) 1_\sigma$,

hold for all $\sigma, \tau \in G$ and $x \in S$.

Proof. The relation (i) follows from the equality $\alpha_\sigma(S_{\sigma^{-1}} \cap S_\tau) = S_\sigma \cap S_{\sigma\tau}$.

Using this relation we have

$$\begin{aligned}
\alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}})) &= \alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}}1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}})) \\
&= \alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}})\alpha_\tau(1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}})) \\
&= \alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}})1_\tau 1_{\sigma^{-1}})) \\
&= \alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}})1_{\sigma^{-1}})
\end{aligned}$$

On the other hand, since $x1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}} \in S_{\tau^{-1}} \cap S_{\tau^{-1}\sigma^{-1}} = \alpha_{\tau^{-1}}(S_\tau \cap S_{\sigma^{-1}})$, using again (i) we have

$$\begin{aligned}
\alpha_\sigma(\alpha_\tau(x1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}})) &= \alpha_{\sigma\tau}(x1_{\tau^{-1}}1_{\tau^{-1}\sigma^{-1}}) \\
&= \alpha_{\sigma\tau}(x1_{\tau^{-1}\sigma^{-1}}1_{\tau^{-1}\sigma^{-1}}1_{\tau^{-1}}) \\
&= \alpha_{\sigma\tau}(x1_{\tau^{-1}\sigma^{-1}})\alpha_{\sigma\tau}(1_{\tau^{-1}\sigma^{-1}}1_{\tau^{-1}}) \\
&= \alpha_{\sigma\tau}(x1_{\tau^{-1}\sigma^{-1}})1_{\sigma\tau}1_\sigma \\
&= \alpha_{\sigma\tau}(x1_{\tau^{-1}\sigma^{-1}})1_\sigma
\end{aligned}$$

□

Given a partial action α of G on the R-algebra S, the skew group ring $S \star_\alpha G$ is defined as the set of all the formal sums $\sum_{\sigma\in G} x_\sigma u_\sigma$, $x_\sigma \in S_\sigma$, with the usual addition and the multiplication determined by $(x_\sigma u_\sigma)(y_\tau u_\tau) = \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)u_{\sigma\tau}$. Since each S_σ is an algebra with identity, then $S \star_\alpha G$ is an associative R-algebra ([5], Corollary 3.2).

A natural map $j : S \star_\alpha G \to \mathrm{End}_R(S)$ is defined by $j(\sum_{\sigma\in G} x_\sigma u_\sigma)(z) = \sum_{\sigma\in G} x_\sigma \alpha_\sigma(z1_{\sigma^{-1}})$, for every $z \in S$. Clearly, j is a homomorphism of left S-modules. We claim that it is also a homomorphism of R-algebras. Indeed, for any $\sigma, \tau \in G$ and $z \in S$, we have

$$\begin{aligned}
j((x_\sigma u_\sigma)(y_\tau u_\tau))(z) &= j(\alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)u_{\sigma\tau})(z) \\
&= \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)\alpha_{\sigma\tau}(z1_{\tau^{-1}\sigma^{-1}}) \\
&= \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)\alpha_{\sigma\tau}(z1_{\tau^{-1}\sigma^{-1}})1_\sigma.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
j(x_\sigma u_\sigma)j(y_\tau u_\tau)(z) &= x_\sigma\alpha_\sigma(y_\tau\alpha_\tau(z1_{\tau^{-1}})1_{\sigma^{-1}}) \\
&= \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau\alpha_\tau(z1_{\tau^{-1}})1_{\sigma^{-1}}) \\
&= \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)\alpha_\sigma(\alpha_\tau(z1_{\tau^{-1}})1_{\sigma^{-1}}).
\end{aligned}$$

By Lemma 2.1

$$\alpha_\sigma(\alpha_\tau(z1_{\tau^{-1}})1_{\sigma^{-1}}) = \alpha_{\sigma\tau}(z1_{\tau^{-1}\sigma^{-1}})1_\sigma$$

and thus $j((x_\sigma u_\sigma)(y_\tau u_\tau)) = j(x_\sigma u_\sigma)j(y_\tau u_\tau)$, as claimed.

Let M be a left $S \star_\alpha G$-module. We put

$$M^G = \{m \in M : (1_\sigma u_\sigma)m = 1_\sigma m, \text{for all } \sigma \in G\},$$

an R-submodule of M. Note that M is an S-module via the embedding $x \mapsto xu_1$ from S into $S \star_\alpha G$.

The algebra S can be considered as a left $S \star_\alpha G$-module via j, that is, $(x_\sigma u_\sigma)y = j(x_\sigma u_\sigma)(y)$, for all $y \in S$ and $\sigma \in G$. Then the subring of invariants is $S^G = \{x \in S \,|\, \alpha_\sigma(x1_{\sigma^{-1}}) = 1_\sigma x, \text{ for all}\, \sigma \in G\}$. Note that $x \in S^G$ is equivalent to $\alpha_\sigma(xa) = x\alpha_\sigma(a)$, for every $a \in S_{\sigma^{-1}}$, $\sigma \in G$. Since each α_σ is R-linear we have $R \subseteq S^G$.

As in the classical Galois theory the trace map plays an important role. In our case we define it by $\mathrm{tr}_{S/R}(x) = \sum_{\sigma \in G} \alpha_\sigma(x1_{\sigma^{-1}})$, for every $x \in S$. By Lemma 2.1 we have

$$\begin{aligned}\alpha_\tau(\mathrm{tr}_{S/R}(x)1_{\tau^{-1}}) &= \sum_{\sigma\in G} \alpha_\tau(\alpha_\sigma(x1_{\sigma^{-1}})1_{\tau^{-1}}) \\ &= \sum_{\sigma\in G} \alpha_{\tau\sigma}(x1_{\sigma^{-1}\tau^{-1}})1_\tau = \mathrm{tr}_{S/R}(x)1_\tau,\end{aligned}$$

for all $\tau \in G$. Hence $\mathrm{tr}_{S/R}(x) \in S^G$, for all $x \in S$, and so $\mathrm{tr}_{S/R} : S \to S^G$ is an R-linear map.

3. Partial Galois Extensions

We say that two elements σ and τ of G are *strongly distinct*, with respect to the partial action α of G on S (α-strongly distinct, for short), if for any non-zero idempotent $e \in S_\sigma \cup S_\tau$ there exists $x \in S$ such that $\alpha_\sigma(x1_{\sigma^{-1}})e \neq \alpha_\tau(x1_{\tau^{-1}})e$.

Now we prove the following result, corresponding to Theorem 1.3 of [2].

Theorem 3.1 *Let α be a partial action of a (finite) group G on an R-algebra S. Then the following statements are equivalent:*

(i) *S is a finitely generated projective R-module and $j : S\star_\alpha G \to End_R(S)$ is an isomorphism of S-modules and R-algebras.*

(ii) *S is a finitely generated projective R-module and for every left $S\star_\alpha G$-module M the map $\mu : S \otimes M^G \to M$, given by $\mu(x \otimes m) = xm$, is an isomorphism of S-modules.*

(iii) *S is a finitely generated projective R-module and the map $\psi : S\otimes S \to \prod_{\sigma\in G} S_\sigma$, defined by $\psi(x \otimes y) = (x\alpha_\sigma(y1_{\sigma^{-1}}))_{\sigma\in G}$, is an isomorphism of S-algebras.*

(iv) *$S^G = R$ and there exist elements $x_i, y_i \in S$, $1 \le i \le n$, such that $\sum_{1\le i\le n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$, for each $\sigma \in G$.*

(v) *$S^G = R$, S is separable over R and the elements of G are pairwise α-strongly distinct.*

Proof. (i)⇒(ii) Since S is a finitely generated projective R-module there exist elements $x_i \in S$ and $f_i \in \mathrm{Hom}_R(S, R)$, $1 \le i \le l$, such that $x = \sum_{1\le i\le l} f_i(x)x_i$, for all $x \in S$.

For each $1 \le i \le l$ put $v_i = j^{-1}(f_i) \in S \star_\alpha G$. Note that $j(1_\sigma u_\sigma v_i)(x) = j(1_\sigma u_\sigma)(j(v_i)(x)) = j(1_\sigma u_\sigma)(f_i(x)) = \alpha_\sigma(f_i(x)1_{\sigma^{-1}}) = f_i(x)1_\sigma = 1_\sigma j(v_i)(x) = j(1_\sigma v_i)(x)$, for every $x \in S$. So $1_\sigma u_\sigma v_i = 1_\sigma v_i$, for every $\sigma \in G$, and consequently $v_i m \in M^G$, for all $m \in M$ and $1 \le i \le l$. This implies that the map $\nu : M \to S \otimes M^G$, defined by $\nu(m) = \sum_{1\le i\le l} x_i \otimes v_i m$, is a well defined homomorphism of R-modules.

Note that $j(\sum_{1\le i\le l} x_i v_i)(x) = \sum_{1\le i\le l} x_i j(v_i)(x) = \sum_{1\le i\le l} x_i f_i(x) = x$, for all $x \in S$, which implies that $\sum_{1\le i\le l} x_i v_i = u_1$.

Also, for $x \in S$, $v = \sum_{\sigma\in G} y_\sigma u_\sigma \in S \star_\alpha G$ and $m \in M^G$ we have

$$\begin{aligned} v(xm) &= (v(xu_1))m = ((\sum_{\sigma\in G} y_\sigma u_\sigma)(xu_1))m \\ &= (\sum_{\sigma\in G} \alpha_\sigma(\alpha_{\sigma^{-1}}(y_\sigma)x)u_\sigma)m = \sum_{\sigma\in G} \alpha_\sigma(\alpha_{\sigma^{-1}}(y_\sigma)x1_{\sigma^{-1}})m \\ &= \sum_{\sigma\in G} y_\sigma\alpha_\sigma(x1_{\sigma^{-1}})m = j(v)(x)m. \end{aligned}$$

Using the above relations we immediately obtain that $\mu\nu$ and $\nu\mu$ are the identity mappings.

(ii)⇒(iii) Put $\mathcal{F} = \{f : G \to S \mid f(\sigma) \in S_\sigma, \text{ for all}\, \sigma \in G\}$. Then it is clear that $\mathcal{F}$ is an S-algebra evidently isomorphic to $\prod_{\sigma\in G} S_\sigma$. For each $\sigma \in G$ and $f \in \mathcal{F}$ the map $\sigma * f : G \to S$ defined by $(\sigma * f)(\tau) = \alpha_\sigma(f(\sigma^{-1}\tau)1_{\sigma^{-1}})$, for every $\tau \in G$, belongs to $\mathcal{F}$. In fact, $(\sigma * f)(\tau) = \alpha_\sigma(f(\sigma^{-1}\tau)1_{\sigma^{-1}\tau}1_{\sigma^{-1}}) = \alpha_\sigma(f(\sigma^{-1}\tau)1_{\sigma^{-1}})\alpha_\sigma(1_{\sigma^{-1}\tau}1_{\sigma^{-1}}) = \alpha_\sigma(f(\sigma^{-1}\tau)1_{\sigma^{-1}})1_\tau 1_\sigma \in S_\tau$. Also, Lemma 2.1 easily implies that $\mathcal{F}$ has a structure of a left $S \star_\alpha G$-module given by $(x_\sigma u_\sigma)f = x_\sigma(\sigma * f)$.

It follows from the assumption (ii) that the map $\mu : S \otimes \mathcal{F}^G \to \mathcal{F} \simeq \prod_{\sigma\in G} S_\sigma$ defined by $\mu(x \otimes f) = (xf(\sigma))_{\sigma\in G}$ is an isomorphism of S-algebras.

Finally, for $x \in S$ denote by f_x the map from G to S defined by $f_x(\tau) = \alpha_\tau(x1_{\tau^{-1}})$, $\tau \in G$. Lemma 2.1 gives $((1_\sigma u_\sigma)f_x)(\tau) = \alpha_\sigma(f_x(\sigma^{-1}\tau)1_{\sigma^{-1}}) = \alpha_\sigma(\alpha_{\sigma^{-1}\tau}(x1_{\tau^{-1}\sigma})1_{\sigma^{-1}}) = \alpha_\tau(x1_{\tau^{-1}})1_\sigma = 1_\sigma f_x(\tau)$, for all $\sigma, \tau \in G$, which means that $f_x \in \mathfrak{F}^G$. The converse is also true: if $f \in \mathfrak{F}^G$ we have $\alpha_\sigma(f(\sigma^{-1}\tau)1_{\sigma^{-1}}) = 1_\sigma f(\tau)$, for all $\sigma, \tau \in G$. In particular, taking $\tau = \sigma$ we obtain $\alpha_\sigma(f(1)1_{\sigma^{-1}}) = 1_\sigma f(\sigma) = f(\sigma)$, and so f is uniquely determined by $f(1)$. These considerations easily imply that the map $S \to \mathfrak{F}^G$, $x \mapsto f_x$, is an isomorphism of S-algebras and consequently the composition $S \otimes S \to S \otimes \mathfrak{F}^G \to \prod_{\sigma \in G} S_\sigma$ is also an isomorphism of S-algebras, which is clearly equal to ψ.

(iii)⇒(iv) Take $x \in S^G$. It follows from (iii) that $x \otimes 1 = 1 \otimes x$. On the other hand, since S is a faithfully projective R-module there exists an R-submodule M of S such that $S = R \oplus M$. Write $x = r + y$ with $r \in R$ and $y \in M$. Then $y \otimes 1 = 1 \otimes y$ and they are in different direct summands of $S \otimes S$. Hence $y \otimes 1 = 0$ and so $y = 0$. Consequently $x = r \in R$ and $S^G = R$ follows.

For the second part of (iv) take $(1, 0, \ldots, 0) \in \prod_{\sigma \in G} S_\sigma$, whose first entry corresponds to $\sigma = 1$. Since ψ is an isomorphism, there exists $w = \sum_{1 \le i \le n} x_i \otimes y_i \in S \otimes S$ such that $\psi(w) = (1, 0, \ldots, 0)$ and (iv) follows.

(iv)⇒(i) Note that $\mathrm{tr}_{S/R}(x) \in R$ for any $x \in S$, since $S^G = R$. Take $x_i, y_i \in S$, $1 \le i \le n$, with $\sum_{1 \le i \le n} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$ and define $f_i \in \mathrm{Hom}_R(S, R)$ by $f_i(x) = \mathrm{tr}_{S/R}(y_i x)$, for all $x \in S$. Then

$$\begin{aligned}
\sum_{1 \le i \le n} f_i(x) x_i &= \sum_{1 \le i \le n} \sum_{\sigma \in G} \alpha_\sigma(y_i x 1_{\sigma^{-1}}) x_i \\
&= \sum_{\sigma \in G} \Big(\sum_{1 \le i \le n} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) \Big) \alpha_\sigma(x 1_{\sigma^{-1}}) \\
&= \sum_{\sigma \in G} \delta_{1,\sigma} \alpha_\sigma(x 1_{\sigma^{-1}}) = x,
\end{aligned}$$

for all $x \in S$. Hence S is a finitely generated projective R-module.

Now we show that j is an isomorphism. For arbitrary $h \in \mathrm{End}_R(S)$ set $w = \sum_{\sigma \in G} \sum_{1 \le i \le n} h(x_i) \alpha_\sigma(y_i 1_{\sigma^{-1}}) u_\sigma \in S \star_\alpha G$. Then for any $x \in S$,

$$\begin{aligned}
j(w)(x) &= \sum_{\sigma \in G} \sum_{1 \le i \le n} h(x_i) \alpha_\sigma(y_i 1_{\sigma^{-1}}) \alpha_\sigma(x 1_{\sigma^{-1}}) \\
&= \sum_{1 \le i \le n} h(x_i) \Big(\sum_{\sigma \in G} \alpha_\sigma(y_i x 1_{\sigma^{-1}}) \Big) = \sum_{1 \le i \le n} h(x_i) \mathrm{tr}_{S/R}(y_i x) \\
&= h\Big(\sum_{1 \le i \le n} x_i \mathrm{tr}_{S/R}(y_i x) \Big) = h\Big(\sum_{\sigma \in G} \sum_{1 \le i \le n} x_i \alpha_\sigma(y_i x 1_{\sigma^{-1}}) \Big) =
\end{aligned}$$

$$
\begin{aligned}
&= h(\sum_{\sigma\in G}(\sum_{1\le i\le n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}))\alpha_\sigma(x1_{\sigma^{-1}}))\\
&= h(\sum_{\sigma\in G}\delta_{1,\sigma}\alpha_\sigma(x1_{\sigma^{-1}})) = h(x),
\end{aligned}
$$

hence j is surjective.

Take now $w = \sum_{\sigma\in G} x_\sigma u_\sigma \in \mathrm{Ker}(j)$. Then $j(w)(x_i) = 0$, for all $1 \le i \le n$. Using Lemma 2.1 we obtain

$$
\begin{aligned}
0 &= \sum_{\tau\in G}\sum_{1\le i\le n} j(w)(x_i)\alpha_\tau(y_i 1_{\tau^{-1}})u_\tau\\
&= \sum_{\tau\in G}\sum_{1\le i\le n}\sum_{\sigma\in G} x_\sigma\alpha_\sigma(x_i 1_{\sigma^{-1}})\alpha_\tau(y_i 1_{\tau^{-1}})u_\tau\\
&= \sum_{\tau\in G}\sum_{\sigma\in G} x_\sigma\alpha_\sigma(\sum_{1\le i\le n} x_i\alpha_{\sigma^{-1}}(\alpha_\tau(y_i 1_{\tau^{-1}})1_\sigma))u_\tau\\
&= \sum_{\tau\in G}\sum_{\sigma\in G} x_\sigma\alpha_\sigma(\sum_{1\le i\le n} x_i\alpha_{\sigma^{-1}\tau}(y_i 1_{\tau^{-1}\sigma})1_{\sigma^{-1}})u_\tau\\
&= \sum_{\tau\in G}\sum_{\sigma\in G} x_\sigma\alpha_\sigma(\delta_{1,\tau^{-1}\sigma}1_{\sigma^{-1}})u_\tau\\
&= \sum_{\sigma\in G} x_\sigma u_\sigma = w
\end{aligned}
$$

and so j is injective.

(iv)$\Rightarrow$(v) If $x_i, y_i \in S$, $1 \le i \le n$, are such that $\sum_{1\le i\le n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$, for all $\sigma \in G$, then the element $e = \sum_{1\le i\le n} x_i \otimes y_i \in S\otimes S$ is the separability idempotent of $S\otimes S$. In fact, $\sum_{1\le i\le n} x_i y_i = 1$ and, for any $x \in S$, we have

$$
\begin{aligned}
\sum_{1\le i\le n} xx_i\otimes y_i &= \sum_{1\le i\le n}(\sum_{\sigma\in G}\alpha_\sigma(xx_i 1_{\sigma^{-1}})\delta_{1,\sigma})\otimes y_i\\
&= \sum_{1\le i\le n}(\sum_{\sigma\in G}\alpha_\sigma(xx_i 1_{\sigma^{-1}})\sum_{1\le j\le n} x_j\alpha_\sigma(y_j 1_{\sigma^{-1}}))\otimes y_i\\
&= \sum_{1\le i,j\le n}(\sum_{\sigma\in G}\alpha_\sigma(xx_i y_j 1_{\sigma^{-1}}))x_j\otimes y_i\\
&= \sum_{1\le i,j\le n} x_j \mathrm{tr}_{S/R}(xx_i y_j)\otimes y_i = \sum_{1\le i,j\le n} x_j\otimes \mathrm{tr}_{S/R}(xx_i y_j)y_i\\
&= \sum_{1\le j\le n} x_j\otimes\sum_{1\le i\le n}\sum_{\sigma\in G}\alpha_\sigma(xx_i y_j 1_{\sigma^{-1}})y_i\\
&= \sum_{1\le j\le n} x_j\otimes\sum_{\sigma\in G}\alpha_\sigma(\sum_{1\le i\le n} x_i\alpha_{\sigma^{-1}}(y_i 1_\sigma)xy_j 1_{\sigma^{-1}})\\
&= \sum_{1\le j\le n} x_j\otimes\sum_{\sigma\in G}\alpha_\sigma(\delta_{1,\sigma}xy_j 1_{\sigma^{-1}}) = \sum_{1\le j\le n} x_j\otimes xy_j.
\end{aligned}
$$

Now take $\sigma,\tau \in G$ and suppose that $v \in S_\sigma \cup S_\tau$ is a non-zero idempotent. We can assume $v \in S_\sigma$. If $\alpha_\sigma(x1_{\sigma^{-1}})v = \alpha_\tau(x1_{\tau^{-1}})v$, for all $x \in S$, using

Lemma 2.1 we have

$$\begin{aligned} x\alpha_{\sigma^{-1}}(v1_\sigma) &= \alpha_{\sigma^{-1}}(\alpha_\tau(x1_{\tau^{-1}})v1_\sigma) \\ &= \alpha_{\sigma^{-1}}(\alpha_\tau(x1_{\tau^{-1}})1_\sigma)\alpha_{\sigma^{-1}}(v1_\sigma) \\ &= \alpha_{\sigma^{-1}\tau}(x1_{\tau^{-1}\sigma})\alpha_{\sigma^{-1}}(v1_\sigma), \end{aligned}$$

for all $x \in S$. In particular we have $\alpha_{\sigma^{-1}}(v1_\sigma) = \sum_{1\le i\le n} x_i y_i \alpha_{\sigma^{-1}}(v1_\sigma) = \sum_{1\le i\le n} x_i \alpha_{\sigma^{-1}\tau}(y_i 1_{\tau^{-1}\sigma})\alpha_{\sigma^{-1}}(v1_\sigma) = \delta_{1,\sigma^{-1}\tau}\alpha_{\sigma^{-1}}(v1_\sigma)$. Since $v1_\sigma \neq 0$ we conclude that $\sigma = \tau$. So the elements of G are pairwise α-strongly distinct.

(v)⇒(iv) For $\sigma \in G$ consider $\theta_\sigma : S\otimes S \to S\otimes S_\sigma$ the homomorphism of S-algebras defined by $\theta_\sigma(x\otimes y) = x\otimes\alpha_\sigma(y1_{\sigma^{-1}})$. Denote by $e = \sum_{1\le i\le n} x_i\otimes y_i \in S\otimes S$ the idempotent of the separability of S over R and by $\mu : S\otimes S \to S$ the multiplication map. It is easy to verify that, for each $\sigma \in G$, the element $e_\sigma = \mu(\theta_\sigma(e)) = \sum x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) \in S_\sigma$ is an idempotent. Furthermore, since S is an $S\otimes S$-module, μ is $S\otimes S$-linear and θ_σ is $S\otimes S^G$-linear, for any $x \in S$ we have

$$\begin{aligned} xe_\sigma &= x\mu(\theta_\sigma(e)) = (x\otimes 1)\mu(\theta_\sigma(e)) = \mu(\theta_\sigma((x\otimes 1)e)) \\ &= \mu(\theta_\sigma((1\otimes x)e)) = \mu(\theta_\sigma(1\otimes x))\mu(\theta_\sigma(e)) \\ &= \alpha_\sigma(x1_{\sigma^{-1}})e_\sigma. \end{aligned}$$

Since the elements of G are pairwise α-strongly distinct, if $\sigma \neq 1$ then $e_\sigma = 0$ and so $\sum_{1\le i\le n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = 0$. The proof is complete. □

We say that S is a *partial Galois extension of R with group G and partial action α* (an *α-partial Galois extension*, for short) if the equivalent conditions of Theorem 3.1 are satisfied. When this is the case, the elements $x_i, y_i \in S$, $1 \le i \le n$, satisfying the conditions of (iv) are called a *Galois coordinate system.*

Clearly, in the particular case that $S_\sigma = S$ for all $\sigma \in G$, this notion of partial Galois extension coincides with the classical notion of Galois extension given in [2].

Corollary 3.2 *Let S be an α-partial Galois extension of R. If $1_G = \prod_{\sigma\in G} 1_\sigma \neq 0$, then $S1_G$ is a (global) Galois extension of $R1_G$ with Galois group G.*

Proof. Obviously, $x1_G \in S_\sigma$, for all $x \in S$ and $\sigma \in G$. Moreover, using Lema 2.1 we easily see that for any $x \in S$, $\alpha_\sigma(x1_G) = \alpha_\sigma(x1_{\sigma^{-1}})1_G$ and for any $y \in S$ we have $y1_G = y1_\sigma 1_G = \alpha_\sigma(x1_{\sigma^{-1}})1_G = \alpha_\sigma(x1_G)$, for some $x \in S$.

So $\alpha_\sigma(S1_G) = S1_G$ and consequently $\alpha_\sigma|_{S1_G} \in \mathrm{Aut}_{R1_G}(S1_G)$, for all $\sigma \in G$. Also $(S1_G)^G \subseteq S^G \cap S1_G = R \cap S1_G \subseteq R1_G \subseteq (S1_G)^G$ and so $(S1_G)^G = R1_G$.

Finally, by the assumption there exist $x_i, y_i \in S$, $1 \leq i \leq n$, with $\sum_{1\leq i\leq n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$, for all $\sigma \in G$. So, $\sum_{1\leq i\leq n}(x_i 1_G)\alpha_\sigma(y_i 1_G) = \sum_{1\leq i\leq n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}})1_G = \delta_{1,\sigma}1_G$, which completes the proof. □

Proposition 3.3 *Let S be an α-partial Galois extension of R. Then we have:*

(i) *There exists $c \in S$ with $\mathrm{tr}_{S/R}(c) = 1$.*

(ii) *For any commutative R-algebra T, $T \otimes S$ is a $1 \otimes \alpha$-partial Galois extension of $T = T \otimes R$, where the partial action of G on $T \otimes S$ is given by the maps $1 \otimes \alpha_\sigma : T \otimes S_{\sigma^{-1}} \to T \otimes S_\sigma$, for any $\sigma \in G$.*

Proof. (i) We claim that for any $f \in \mathrm{Hom}_R(S, R)$ there exists $s = s(f) \in S$ such that $f(y) = \mathrm{tr}_{S/R}(sy)$, for all $y \in S$.

In fact, since $\mathrm{Hom}_R(S,R) \subseteq \mathrm{End}_R(S) \overset{j^{-1}}{\simeq} S \star_\alpha G$, there exists $w = \sum_{\sigma\in G} x_\sigma u_\sigma \in S \star_\alpha G$ such that $f = j(w)$. This means that $f(y) = j(w)(y) = \sum_{\sigma\in G} x_\sigma \alpha_\sigma(y1_{\sigma^{-1}})$, for all $y \in S$. Using Lemma 2.1 we have

$$\begin{aligned}\alpha_\tau(f(y)1_{\tau^{-1}}) &= \alpha_\tau(\sum_{\sigma\in G} x_\sigma\alpha_\sigma(y1_{\sigma^{-1}})1_{\tau^{-1}}) \\ &= \sum_{\sigma\in G}\alpha_\tau(x_\sigma 1_{\tau^{-1}})\alpha_{\tau\sigma}(y1_{\sigma^{-1}\tau^{-1}}) \\ &= \sum_{\rho\in G}\alpha_\tau(x_{\tau^{-1}\rho}1_{\tau^{-1}})\alpha_\rho(y1_{\rho^{-1}}),\end{aligned}$$

for all $\tau \in G$.

On the other hand, since $f(y) \in R = S^G$ we have $\alpha_\tau(f(y)1_{\tau^{-1}}) = f(y)1_\tau$. So $\sum_{\rho\in G}\alpha_\tau(x_{\tau^{-1}\rho}1_{\tau^{-1}})\alpha_\rho(y1_{\rho^{-1}}) = \sum_{\rho\in G} x_\rho\alpha_\rho(y1_{\rho^{-1}})1_\tau$ and this implies that $j(\sum_{\rho\in G}\alpha_\tau(x_{\tau^{-1}\rho}1_{\tau^{-1}})1_\rho u_\rho)(y) = j(\sum_{\rho\in G} x_\rho 1_\tau u_\rho)(y)$, for all $y \in S$ and $\tau \in G$. Consequently $\sum_{\rho\in G}\alpha_\tau(x_{\tau^{-1}\rho}1_{\tau^{-1}})1_\rho u_\rho = \sum_{\rho\in G} x_\rho 1_\tau u_\rho$ and thus $\alpha_\tau(x_{\tau^{-1}\rho}1_{\tau^{-1}})1_\rho = x_\rho 1_\tau$, for every $\rho, \tau \in G$. In particular, for $\rho = \tau$ one has $\alpha_\tau(x_1 1_{\tau^{-1}}) = x_\tau$, for all $\tau \in G$. Hence for $y \in S$, $f(y) = \sum_{\tau\in G} x_\tau\alpha_\tau(y1_{\tau^{-1}}) = \sum_{\tau\in G}\alpha_\tau(x_1 1_{\tau^{-1}})\alpha_\tau(y1_{\tau^{-1}}) = \sum_{\tau\in G}\alpha_\tau(x_1 y 1_{\tau^{-1}}) = \mathrm{tr}(x_1 y)$, as claimed.

Now, since S is a faithfully projective R-module, there exist $y_i \in S$ and $f_i \in \mathrm{Hom}_R(S,R)$, $1 \leq i \leq n$, with $\sum_{1\leq i\leq n} f_i(y_i) = 1$. Take $x_i \in S$ such that $f_i(y) = \mathrm{tr}_{S/R}(x_i y)$, for all $y \in S$. Then $1 = \sum_{1\leq i\leq n} f_i(y_i) = \sum_{1\leq i\leq n}\mathrm{tr}_{S/R}(x_i y_i) = \mathrm{tr}_{S/R}(\sum_{1\leq i\leq n} x_i y_i)$ and the result follows.

(ii) It is easy to see that $(T \otimes S_\sigma, 1 \otimes \alpha_\sigma)$ defines a partial action on $T \otimes S$. Also T can be identified with $T \otimes R \subseteq T \otimes S$. Furthermore, a Galois coordinate system for S over R easily gives a Galois coordinate system for $T \otimes S$ over $(T \otimes S)^G$. It remains to show that $T = T \otimes R = (T \otimes S)^G$.

Evidently $T \otimes R \subseteq (T \otimes S)^G$. Conversely take $w \in (T \otimes S)^G$. Then if $c \in S$ is such that $\mathrm{tr}_{S/R}(c) = 1$ one has $\sum_{\sigma \in G}(1 \otimes \alpha_\sigma)((1 \otimes c)(1 \otimes 1_{\sigma^{-1}})) = 1 \otimes \sum_{\sigma \in G} \alpha_\sigma(c 1_{\sigma^{-1}}) = 1 \otimes 1$ and therefore

$$
\begin{aligned}
w &= w(1 \otimes 1) = w \sum_{\sigma \in G} (1 \otimes \alpha_\sigma)((1 \otimes c)(1 \otimes 1_{\sigma^{-1}})) \\
&= \sum_{\sigma \in G} (1 \otimes \alpha_\sigma)(w(1 \otimes 1_{\sigma^{-1}}))(1 \otimes \alpha_\sigma)((1 \otimes c)(1 \otimes 1_{\sigma^{-1}})) \\
&= \sum_{\sigma \in G} (1 \otimes \alpha_\sigma)(w(1 \otimes c)(1 \otimes 1_{\sigma^{-1}})).
\end{aligned}
$$

Writing $w(1 \otimes c) = \sum_{1 \le i \le m} t_i \otimes s_i \in T \otimes S$ we have that

$$
\begin{aligned}
w &= \sum_{1 \le i \le m} \sum_{\sigma \in G} (1 \otimes \alpha_\sigma)(t_i \otimes s_i 1_{\sigma^{-1}}) \\
&= \sum_{1 \le i \le m} t_i \otimes (\sum_{\sigma \in G} \alpha_\sigma(s_i 1_{\sigma^{-1}})) \\
&= \sum_{1 \le i \le m} t_i \otimes \mathrm{tr}_{S/R}(s_i) \in T \otimes R,
\end{aligned}
$$

and the proof is complete. □

As a consequence of Theorem 3.1 and Proposition 3.3 we obtain the following

Corollary 3.4 *Let S be an α-partial Galois extension of R. Then, for any prime ideal $\wp$ of R, $rank_{R_\wp}(S_\wp) \le |G|$. Moreover, $rank_{R_\wp}(S_\wp) = |G|$ for every $\wp$ if and only if S is a (global) Galois extension of R with group G.*

Proof. By Proposition 3.3 (ii) we can assume that R is a local ring. So S and S_σ are finitely generated free R-modules and $S \otimes S \simeq \prod_{\sigma \in G} S_\sigma$, by Theorem 3.1 (iii). Consequently,

$$(rank_R(S))^2 = rank_R(S \otimes S) = \sum_{\sigma \in G} rank_R(S_\sigma) \le |G| rank_R(S)$$

and the first part follows.

For the second assertion note, in particular, that the above relation implies $rank_R(S) = |G|$ if and only if $rank_R(S_\sigma) = rank_R(S)$, for every $\sigma \in G$. On the other hand, since S is separable over R, so is S_σ and by ([14], Lemma

1.1) we obtain $rank_R(S_\sigma) = rank_R(S)$ if and only if $S_\sigma = S$. The result follows. □

4. Enveloping Galois actions

Assume that α is a partial action of G on S, as given in the former section, i.e., any of the ideals S_σ has identity element 1_σ and $S^G = R$.

By Theorem 4.5 of [5], α possesses an enveloping action, which means that there exists a ring S' and a (global) action of G by automorphisms of S' such that S can be considered as an ideal of S' and the following properties hold:

(i) the subalgebra of S' generated by $\bigcup_{\sigma\in G}\sigma(S)$ coincides with S' and we have $S' = \sum_{\sigma\in G}\sigma(S)$,

(ii) $S_\sigma = S\bigcap\sigma(S)$, for every $\sigma \in G$,

(iii) $\alpha_\sigma(x) = \sigma(x)$, for all $\sigma \in G$ and $x \in S_{\sigma^{-1}}$.

In the rest of the paper we will say that (S', G) is an *enveloping action* of α.

Note that $S'1_\sigma = S1_\sigma = S\bigcap\sigma(S) = S'1_S\bigcap S'\sigma(1_S) = S'1_S\sigma(1_S)$, and this implies that $1_\sigma = 1_S\sigma(1_S)$, where 1_S denotes the identity element of S. This remark will be used frequently in the sequel.

Set $R' = (S')^G = \{x \in S' \mid \sigma(x) = x, \text{ for all}\, \sigma \in G\}$. Then S' is an extension of R' and we may assume $G \subseteq \mathrm{Aut}_{R'}(S')$.

Theorem 4.1 *Let R, S, R', S', G and α as above. Then the following statements are equivalent*

(i) *S' is a (global) Galois extension of R' with Galois group G.*

(ii) *S is an α-partial Galois extension of R.*

Proof. (i)⇒(ii) By assumption $R = S^G$. Let $x'_i, y'_i \in S'$, $1 \le i \le m$, such that $\sum_{1\le i\le m} x'_i\sigma(y'_i) = \delta_{1,\sigma}1_{S'}$ for every $\sigma \in G$, and consider the elements $x_i = x'_i1_S$ and $y_i = y'_i1_S$, $1 \le i \le m$. Then $\sum_{1\le i\le m} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = \sum_{1\le i\le m} x'_i1_S\sigma(y'_i1_S1_{\sigma^{-1}}) = \sum_{1\le i\le m} x'_i\sigma(y'_i)1_S\sigma(1_S)1_\sigma = \sum_{1\le i\le m} x'_i\sigma(y'_i)1_\sigma = \delta_{1,\sigma}1_\sigma = \delta_{1,\sigma}1_S$, for all $\sigma \in G$, and the result follows.

(ii)⇒(i) Put $G = \{\sigma_1 = 1, \sigma_2, \ldots, \sigma_n\}$. It is easy to see that the identity element of S' is given by

$$1_{S'} = \sum_j \sigma_j(1_S) - \sum_{j<k} \sigma_j(1_S)\sigma_k(1_S) + \cdots + (-1)^{n+1}\prod_j \sigma_j(1_S).$$

Each term of this sum is an idempotent element of S' and we can write $1_{S'}$ as an orthogonal sum $1_{S'} = e_1 \oplus e_2 \oplus \cdots \oplus e_n$, where $e_1 = 1_S$, $e_2 = (1_{S'} - 1_S)\sigma_2(1_S)$ and $e_j = (1_{S'} - 1_S)\ldots(1_{S'} - \sigma_{j-1}(1_S))\sigma_j(1_S)$, for $2 \le j \le n$.

By assumption there exist $x_i, y_i \in S$, $1 \le i \le m$, such that $\sum_{1\le i\le m} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}1_S$, for every $\sigma \in G$. Consider in S' the elements $x'_{ij} = \sigma_j(x_i)e_j$ and $y'_{ij} = \sigma_j(y_i)e_j$, $1 \le i \le m$, $1 \le j \le n$. Then

$$\begin{aligned}\sum_{i,j} x'_{ij}y'_{ij} &= \sum_{i,j} \sigma_j(x_i)e_j\sigma_j(y_i)e_j = \sum_j \sigma_j(\sum_i x_iy_i)e_j \\ &= \sum_j \sigma_j(1_S)e_j = \sum_j e_j = 1_{S'}.\end{aligned}$$

Also, putting $g_j = (1_{S'} - 1_S)\ldots(1_{S'} - \sigma_{j-1}(1_S))$ we have

$$\begin{aligned}\sum_{i,j} x'_{ij}\sigma_l(y'_{ij}) &= \sum_{i,j} \sigma_j(x_i)\sigma_l(\sigma_j(y_i))e_j\sigma_l(e_j) \\ &= \sum_i x_i\sigma_l(y_i)1_S\sigma_l(1_S) + \\ &\quad \sum_{j>1}\sum_i \sigma_j(x_i)\sigma_l(\sigma_j(y_i))\sigma_j(1_S)\sigma_l(\sigma_j(1_S))g_j\sigma_l(g_j) \\ &= \sum_i x_i\sigma_l(y_i)1_{\sigma_l} + \\ &\quad \sum_{j>1} \sigma_j(\sum_i x_i\sigma_j^{-1}\sigma_l\sigma_j(y_i)1_S\sigma_j^{-1}\sigma_l\sigma_j(1_S))g_j\sigma_l(g_j) \\ &= \sum_i x_i\alpha_{\sigma_l}(y_i1_{\sigma_l^{-1}}) + \\ &\quad \sum_{j>1} \sigma_j(\sum_i x_i\alpha_{\sigma_j^{-1}\sigma_l\sigma_j}(y_i1_{(\sigma_j^{-1}\sigma_l\sigma_j)^{-1}}))g_j\sigma_l(g_j) = 0,\end{aligned}$$

for $2 \le l \le n$, and the proof is complete. $\square$

In the rest of the section we use the same notation as above and we assume that the equivalent conditions of Theorem 3.1 are satisfied. We study the relation between $R = S^G$ and $R' = (S')^G$.

For any $r \in R$ we put

$$\psi(r) = \sum_{1\le l\le n}\sum_{i_1<\cdots<i_l} (-1)^{l+1}\sigma_{i_l}(r)\sigma_{i_1}(1_S)\ldots\sigma_{i_l}(1_S).$$

Note that $\sum_{\sigma\in G}\alpha_\sigma(x1_{\sigma^{-1}}) = \sum_{\sigma\in G}\sigma(x)1_\sigma = \sum_{\sigma\in G}\sigma(x1_S)1_S = \sum_{\sigma\in G}\sigma(x)1_S$, so we have $\mathrm{tr}_{S/R}(x) = \mathrm{tr}_{S'/R'}(x)1_S$, for any $x \in S$. Since $R = \mathrm{tr}_{S/R}(S)$, given $r \in R$ there exists $x \in S$ with $\mathrm{tr}_{S/R}(x) = r$. Hence $r = \mathrm{tr}_{S'/R'}(x)1_S$ and

$$\begin{aligned}\psi(r) &= \mathrm{tr}_{S'/R'}(x)(\sum_{1\le l\le n}\sum_{i_1<\cdots<i_l}(-1)^{l+1}\sigma_{i_1}(1_S)\ldots\sigma_{i_l}(1_S))\\ &= \mathrm{tr}_{S'/R'}(x)1_{S'} = \mathrm{tr}_{S'/R'}(x).\end{aligned}$$

Thus we obtain a (well-defined) map $\psi : R \to R'$ given by $\psi(r) = \mathrm{tr}_{S'/R'}(x)$, where x is any element in S such that $\mathrm{tr}_{S/R}(x) = r$.

Now we can prove the following

Theorem 4.2 *Under the above notation we have:*

(i) $R' = \mathrm{tr}_{S'/R'}(S)$,

(ii) $R'1_S = R$,

(iii) ψ *is an isomorphism of rings.*

Proof. (i) Obviously $\mathrm{tr}_{S'/R'}(S) \subseteq \mathrm{tr}_{S'/R'}(S') = R'$. Conversely, let $c \in S'$ be such that $\mathrm{tr}_{S'/R'}(c) = 1_{S'}$. For an arbitrary $x \in R'$ we can write $cx = \sum_{\sigma\in G}\sigma(x_\sigma)$ with $x_\sigma \in S$, since $S' = \sum_{\sigma\in G}\sigma(S)$. Thus we have

$$\begin{aligned}x &= x1_{S'} = x\mathrm{tr}_{S'/R'}(c) = x\sum_{\rho\in G}\rho(c)\\ &= \sum_{\rho\in G}\rho(xc) = \sum_{\sigma\in G}(\sum_{\rho\in G}\rho\sigma(x_\sigma))\\ &= \sum_{\sigma\in G}\mathrm{tr}_{S'/R'}(x_\sigma) \in \mathrm{tr}_{S'/R'}(S).\end{aligned}$$

Hence $R' = \mathrm{tr}_{S'/R'}(S)$.

(ii) As we pointed out above we have $\mathrm{tr}_{S'/R'}(S)1_S = \mathrm{tr}_{S/R}(S)$. Since $\mathrm{tr}_{S/R}(S) = R$ the result follows from (i).

(iii) It is clear that ψ is an additive mapping. Using the same notation as in the proof of Theorem 4.1 we can easily see that $\psi(r) = \sum_{1\le i\le n}\sigma_i(r)e_i$. Since $e_1, \ldots, e_n$ are pairwise orthogonal idempotents it is clear that ψ is multiplicative and injective. Now, take $r' \in R'$ and choose $x \in S$ with $r' = \mathrm{tr}_{S'/R'}(x)$. Then for $r = \mathrm{tr}_{S/R}(x)$ one has $\psi(r) = \mathrm{tr}_{S'/R'}(x) = r'$ and so $\psi(R) = R'$. Finally, since $1_R = 1_S$ and $1_{R'} = 1_{S'}$, we have

$$\psi(1_R) = \sum_{1\le l\le n}\sum_{i_1<\cdots<i_l}(-1)^{l+1}\sigma_{i_1}(1_S)\ldots\sigma_{i_l}(1_S) = 1_{S'} = 1_{R'}. \qquad \square$$

Remark 4.3 From the proof of Theorem 4.2 it easily follows that the inverse of the isomorphism ψ is the map $\psi' : R' \to R$ defined by $\psi'(r') = r'1_S$.

5. The fundamental results

In this section we assume that S is a partial Galois extension of R with group G and partial action α. For a subalgebra T of S we denote by H_T the subgroup of G defined by $H_T = \{\sigma \in G \mid \alpha_\sigma(x1_{\sigma^{-1}}) = x1_\sigma, \text{ for all } x \in T\}$. We say that T is α-strong if for every $\sigma, \tau \in G$, with $\sigma^{-1}\tau \notin H_T$, and any non-zero idempotent $e \in S_\sigma \cup S_\tau$ there exists an element $t \in T$ such that $\alpha_\sigma(t1_{\sigma^{-1}})e \neq \alpha_\tau(t1_{\tau^{-1}})e$. If the action of G on S is a global action, then we have the well-known notion of G-strong subalgebra.

The purpose of this section is to extend Theorem 2.3 of [2] on the one-to-one correspondence between subgroups of G and G-strong separable subalgebras of S, using the results of Section 4. First we prove the following

Theorem 5.1 *Let S be an α-partial Galois extension of R, H a subgroup of G and $T = S^H$. Then*

(i) *S is a partial Galois extension of T with the action $\alpha_H = \{\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma \mid \sigma \in H\}$ of H on S.*

(ii) *T is R-separable and α-strong.*

(iii) *$H_T = H$.*

Proof. Obviously the restriction α_H of α to H is a partial action of H on S.

(i) It follows directly from (iv) of Theorem 3.1.

(ii) Denote by (S', G) the enveloping action of α. Recall that $S' = \sum_{\sigma\in G} \sigma(S)$. It is easy to see that there exists a global action of H on $\widetilde{S} = \sum_{\sigma\in H} \sigma(S)$ which is the enveloping action of α_H.

The global action of G on S' induces a partial action of G on $\widetilde{S}$ for which it is also enveloping. Then Theorem 4.2 implies that $(S')^G 1_{\widetilde{S}} = (\widetilde{S})^G$ and $(S')^G 1_S = S^G$. Consequently $(\widetilde{S})^G 1_S = (S')^G 1_{\widetilde{S}} 1_S = (S')^G 1_S = S^G$. Moreover, $(\widetilde{S})^H 1_S = S^H$.

On the other hand, $S' = S'1_{\widetilde{S}} \oplus S'(1_{S'} - 1_{\widetilde{S}}) = \widetilde{S} \oplus S'(1_{S'} - 1_{\widetilde{S}})$. Clearly, $S'(1_{S'} - 1_{\widetilde{S}})$ is H-invariant, hence $(S')^H = (\widetilde{S})^H \oplus (S'(1_{S'} - 1_{\widetilde{S}}))^H$ and $(S')^H 1_{\widetilde{S}} = (\widetilde{S})^H 1_{\widetilde{S}} = (\widetilde{S})^H$. Thus $T = S^H = (\widetilde{S})^H 1_S = (S')^H 1_{\widetilde{S}} 1_S = (S')^H 1_S$.

By Theorem 4.1 S' is a Galois extension of $R' = (S')^G$ with Galois group G, hence by the results in [2] the R'-algebra $(S')^H$ is separable, G-strong and $H = H_{(S')^H}$. It is easy to verify that if $e' \in (S')^H \otimes_{R'} (S')^H$ is the separability idempotent of $(S')^H$ over R', then $e = e'(1_S \otimes 1_S) \in (S')^H 1_S \otimes_{R'1_S} (S')^H 1_S = S^H \otimes S^H = T \otimes T$ is the separability idempotent of T over R. Thus T is R-separable.

It remains to show that T is α-strong. Take any $\sigma, \tau \in G$ with $\sigma^{-1}\tau \notin H_T$ and a non-zero idempotent $e \in S_\sigma \bigcup S_\tau$. Since $(S')^H$ is G-strong and $\sigma^{-1}\tau \notin H_T \supseteq H = H_{(S')^H}$, if $e1_\sigma 1_\tau \neq 0$ there exists $x \in (S')^H$ such that $\sigma(x)e1_\sigma 1_\tau \neq \tau(x)e1_\sigma 1_\tau$. Consequently $x1_S \in T$ and $\alpha_\sigma(x1_S1_{\sigma^{-1}})e1_\sigma 1_\tau = \alpha_\sigma(x1_{\sigma^{-1}})e1_\sigma 1_\tau = \sigma(x)e1_\sigma 1_\tau \neq \tau(x)e1_\sigma 1_\tau = \alpha_\tau(x1_S1_{\tau^{-1}})e1_\sigma 1_\tau$, and therefore $\alpha_\sigma(x1_S1_{\sigma^{-1}})e \neq \alpha_\tau(x1_S1_{\tau^{-1}})e$. Finally, if $e1_\sigma 1_\tau = 0$ and $e \in S_\sigma$ we have $\alpha_\sigma(1_S1_{\sigma^{-1}})e = 1_\sigma e = e \neq 0 = e1_\sigma 1_\tau = e1_\tau = \alpha_\tau(1_S1_{\tau^{-1}})e$. The case $e \in S_\tau$ is similar and this completes the proof of (ii).

(iii) Obviously $H_T \supseteq H$. Conversely, assume that $\sigma \in H_T \setminus H = H_{(S')^H}$. Since $(S')^H$ is G-strong it follows that there is $x \in (S')^H$ with $\sigma(x)1_\sigma \neq x1_\sigma$. Hence $\alpha_\sigma(x1_S1_{\sigma^{-1}}) = \sigma(x)1_\sigma \neq x1_\sigma = x1_S1_\sigma$ which is a contradiction because $\sigma \in H_T$ and $x1_S \in (S')^H 1_S = T$. Thus $H_T = H$, as desired. □

For the next result we need the following

Lemma 5.2 *Let S be an α-partial Galois extension of R and T a separable and α-strong R-subalgebra of S. Then there exist $x_i, y_i \in T$, $1 \leq i \leq m$, such that $\sum_{1\leq i\leq m} x_i y_i = 1$ and $\sum_{1\leq i\leq m} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) = 0$, for all $\sigma \in G \setminus H_T$.*

Proof. Let $e = \sum_{1\leq i\leq m} x_i \otimes y_i \in T \otimes T$ the separability idempotent of T over R. Then $\sum_{1\leq i\leq m} x_i y_i = 1$ and $\sum_{1\leq i\leq m} xx_i \otimes y_i = \sum_{1\leq i\leq m} x_i \otimes xy_i$, for all $x \in T$. Denote by $\mu : T \otimes T \to T$ the multiplication map. For any $\sigma \in G$ take $\theta_\sigma = \mu \circ 1 \otimes \alpha_\sigma(\ldots 1_{\sigma^{-1}})$ and $f_\sigma = \theta_\sigma(e)$. Then $f_\sigma = \sum_{1\leq i\leq m} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) \in S_\sigma$, $f_\sigma^2 = f_\sigma$ and for every $x \in T$ we have

$$\begin{aligned}
\alpha_\sigma(x1_{\sigma^{-1}})f_\sigma &= \sum_{1\leq i\leq m} x_i \alpha_\sigma(xy_i 1_{\sigma^{-1}}) = \theta_\sigma\Big(\sum_{1\leq i\leq m} x_i \otimes xy_i\Big) \\
&= \theta_\sigma\Big(\sum_{1\leq i\leq m} xx_i \otimes y_i\Big) = \sum_{1\leq i\leq m} xx_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) \\
&= xf_\sigma.
\end{aligned}$$

Since T is α-strong, if $\sigma \notin H_T$ the above implies that $f_\sigma = 0$. □

The next theorem completes the proof of the one-to-one correspondence between subgroups of G and subalgebras of S which are R-separable and α-strong.

Theorem 5.3 *Let S be an α-partial Galois extension of R and T a separable and α-strong R-subalgebra of S. Then $S^H = T$ for $H = H_T$.*

Proof. Clearly $T \subseteq S^H$. So, it remains to prove the converse.

Let (S', G) be the enveloping action of α and $R' = S'^G$. As in the proof of Theorem 5.1 we consider the subalgebra $\widetilde{S} = \sum_{\sigma\in H} \sigma(S)$ on which H acts as a group of automorphisms and $(\widetilde{S}, H)$ is the enveloping action of $\alpha_H = \{(S_\sigma, \alpha_\sigma) \mid \sigma \in H\}$, a partial action of H on S.

By Remark 4.3 there exists a ring isomorphism $\psi_H : S^H \to (\widetilde{S})^H$ such that $\psi_H(x)1_S = x$, for all $x \in S^H$. Write $\widetilde{T} = \psi_H(T)$.

Claim 1 There are elements $\widetilde{x}_i, \widetilde{y}_i \in \widetilde{T}$, $1 \leq i \leq m$, such that $\sum_{1\leq i\leq m} \widetilde{x}_i\widetilde{y}_i = 1$ and $\sum_{1\leq i\leq m} \widetilde{x}_i\sigma(\widetilde{y}_i) = 0$, for every $\sigma \in G \setminus H$.

Indeed, by Lemma 5.2 there exist elements $x_i, y_i \in T$, $1 \leq i \leq m$, such that $\sum_{1\leq i\leq m} x_iy_i = 1$ and $\sum_{1\leq i\leq m} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = 0$, for all $\sigma \in G \setminus H$. Since $\sum_{1\leq i\leq m} x_i\sigma(y_i) \in S \cap \sigma(S) = S_\sigma$ we have that $\sum_{1\leq i\leq m} x_i\sigma(y_i) = \sum_{1\leq i\leq m} x_i\sigma(y_i)1_\sigma = \sum_{1\leq i\leq m} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = 0$.

Write $\widetilde{x}_i = \psi_H(x_i)$ and $\widetilde{y}_i = \psi_H(y_i)$ in $\widetilde{T}$, $1 \leq i \leq m$, and denote by $\tau_1 = 1, \tau_2, \ldots, \tau_l$ the elements of H. As it was seen in the proof of Theorem 4.2 (ii), for every $x \in S^H$ we have $\psi_H(x) = \sum_{1\leq i\leq l} \tau_i(x)e_i$, where $e_1, \ldots, e_l \in \widetilde{S}$ are pairwise orthogonal idempotents. Therefore $\sum_{1\leq i\leq m} \widetilde{x}_i\widetilde{y}_i = \psi_H(\sum_{1\leq i\leq m} x_iy_i) = \psi_H(1_S) = 1_{\widetilde{S}}$ and for each $\sigma \in G \setminus H$ we have

$$\begin{aligned}\textstyle\sum_{1\leq i\leq m} \widetilde{x}_i\sigma(\widetilde{y}_i) &= \sum_{1\leq i\leq m}\sum_{1\leq j,j'\leq l} \tau_j(x_i)e_j\sigma(\tau_{j'}(y_i)e_{j'}) \\ &= \sum_{1\leq j,j'\leq l} e_j\sigma(e_{j'})\tau_j(\sum_{1\leq i\leq m} x_i\tau_j^{-1}\sigma\tau_{j'}(y_i)) = 0,\end{aligned}$$

which completes the proof of Claim 1.

Note that since $H \subseteq H_{\widetilde{T}}$ and the elements $\widetilde{x}_i, \widetilde{y}_i$ of Claim 1 are in $\widetilde{T}$, this claim implies, in particular, that $H_{\widetilde{T}} = H$.

As we saw in Theorem 5.1, the restriction of (S', G) to $\widetilde{S}$ gives a partial action of G on $\widetilde{S}$ for which it is also enveloping. Then it follows from Theorem 4.2 that there is an isomorphism of rings $(S')^G \to (\widetilde{S})^G$ sending x to $x1_{\widetilde{S}}$.

Also, the map $x \mapsto x1_S$ is an isomorphism from $(S')^G$ onto R. Thus we have an isomorphism $(\widetilde{S})^G \to R$ defined by $y \mapsto y1_S$, for any $y \in (\widetilde{S})^G$, whose inverse is ψ_H restricted to R. Hence $\psi_H(R) = (\widetilde{S})^G$ and consequently $\widetilde{T} = \psi_H(T)$ is separable over $(\widetilde{S})^G$.

Note that $(S')^H = (S')^H 1_{\widetilde{S}} \oplus (S')^H(1_{S'} - 1_{\widetilde{S}}) = (S'1_{\widetilde{S}})^H \oplus (S')^H(1_{S'} - 1_{\widetilde{S}}) = (\widetilde{S})^H \oplus (S')^H(1_{S'} - 1_{\widetilde{S}})$. Consider the subalgebra $T' = \widetilde{T} \oplus (S')^H(1_{S'} - 1_{\widetilde{S}})$.

Claim 2 T' is separable over R' and G-strong.

In fact, since $(\widetilde{S})^G$ and $R'(1_{S'} - 1_{\widetilde{S}})$ are separable over R', then $(\widetilde{S})^G \oplus R'(1_{S'} - 1_{\widetilde{S}})$ is R'-separable. Also, S'^H is separable over R' and so $(S')^H(1_{S'} - 1_{\widetilde{S}})$ is separable over $R'(1_{S'} - 1_{\widetilde{S}})$. Because we also have that $\widetilde{T}$ is $(\widetilde{S})^G$-separable, it follows that T' is separable over $(\widetilde{S})^G \oplus R'(1_{S'} - 1_{\widetilde{S}})$ and consequently T' is separable over R'.

To prove that T' is G-strong assume that $\sigma \in G \setminus H$ and $e \in S'$ is an idempotent, and suppose that $\sigma(x + y)e = (x + y)e$, for all $x \in \widetilde{T}$ and $y \in (S')^H(1_{S'} - 1_{\widetilde{S}})$.

Put $e = e_1 + e_2$ with $e_1 = e1_{\widetilde{S}}$ and $e_2 = e(1_{S'} - 1_{\widetilde{S}})$. Then multiplying $\sigma(x + y)e = (x + y)e$ by $1_{\widetilde{S}}$ we obtain $\sigma(x + y)e_1 = (x + y)e_1 = xe_1$, for all $x \in \widetilde{T}$ and $y \in (S')^H(1_{S'} - 1_{\widetilde{S}})$. In particular, taking $y = 0$ we see that $\sigma(x)e_1 = xe_1$, for every $x \in \widetilde{T}$.

By Claim 1 there exist $\widetilde{x}_i, \widetilde{y}_i \in \widetilde{T}$, $1 \leq i \leq m$, with $\sum_{1\leq i\leq m} \widetilde{x}_i\widetilde{y}_i = 1_{\widetilde{S}}$ and $\sum_{1\leq i\leq m} \widetilde{x}_i\sigma(\widetilde{y}_i) = 0$. Hence $0 = \sum_{1\leq i\leq m} \widetilde{x}_i\sigma(\widetilde{y}_i)e_1 = \sum_{1\leq i\leq m} \widetilde{x}_i\widetilde{y}_ie_1 = 1_{\widetilde{S}}e_1 = e_1$.

Thus $e = e_2$ and $\sigma(x + y)e_2 = (x + y)e_2 = ye_2$, for all $x \in \widetilde{T}$ and $y \in (S')^H(1_{S'} - 1_{\widetilde{S}})$. Taking $x = 0$ we obtain $\sigma(y)e_2 = ye_2$, for every $y \in (S')^H(1_{S'} - 1_{\widetilde{S}})$.

On the other hand, since $(S')^H$ is G-strong and separable over R', by Lemma 4.2 there exist $u_j, v_j \in (S')^H$, $1 \leq j \leq l$, with $\sum_{1\leq j\leq l} u_jv_j = 1_{S'}$ and $\sum_{1\leq j\leq l} u_j\sigma(v_j) = 0$. Consequently $0 = \sum_{1\leq j\leq l} u_j\sigma(v_j)\sigma(1_{S'} - 1_{\widetilde{S}})e_2 = \sum_{1\leq j\leq l} u_jv_j(1_{S'} - 1_{\widetilde{S}})e_2 = 1_{S'}(1_{S'} - 1_{\widetilde{S}})e_2 = e_2$, which completes the proof of Claim 2.

Now we are able to complete the proof of the theorem. By Claim 2 and the results in [2] $T' = (S')^{H'}$ for $H' = \{\sigma \in G \mid \sigma(x) = x, \text{ for all } x \in T'\}$. Also, by definition of T', an element $\sigma \in G$ is in $H_{T'} = H'$ if and only if $\sigma \in H_{\widetilde{T}} = H$ and thus $H' = H$. It follows that $T' = (S')^H$ and $S^H =$

$(\widetilde{S})^H 1_S = (S')^H 1_S = T'1_S = \widetilde{T} 1_S = T$. The proof is complete. $\square$

The above results immediately imply the following Fundamental Theorem of partial Galois theory, which is an extension of Theorem 2.3 of [2].

Corollary 5.4 *Let S be a partial Galois extension of R with group G and action α. Then there is a one-to-one correspondence between the subgroups of G and the separable subalgebras of S which are α-strong.*

6. Examples and Remarks

In this section we give some examples and remarks which illustrate our results.

Example 6.1 Let R be a commutative ring and put $S = Re_1 \oplus Re_2 \oplus Re_3$, where $\{e_1, e_2, e_3\}$ is a set of non-zero orthogonal idempotents whose sum is one. We denote by G the cyclic group of order 4 generated by σ, and define a partial action of G on S taking $S_1 = S$, $S_\sigma = Re_1 \oplus Re_2$, $S_{\sigma^2} = Re_1 \oplus Re_3$ and $S_{\sigma^3} = Re_2 \oplus Re_3$, and defining $\alpha_1 = id_S$,

$\alpha_\sigma : S_{\sigma^3} \to S_\sigma$ by $\alpha_\sigma(e_2) = e_1$ and $\alpha_\sigma(e_3) = e_2$,

$\alpha_{\sigma^2} : S_{\sigma^2} \to S_{\sigma^2}$ by $\alpha_{\sigma^2}(e_1) = e_3$ and $\alpha_{\sigma^2}(e_3) = e_1$ and

$\alpha_{\sigma^3} : S_\sigma \to S_{\sigma^3}$ by $\alpha_{\sigma^3}(e_1) = e_2$ and $\alpha_{\sigma^3}(e_2) = e_3$.

Then it is easy to verify that S is an α-partial Galois extension of R. In this case it is also clear that the enveloping action of α is the trivial extension $S' = S \oplus Re_4$ of R, where the global action is given by $\sigma^i(e_j) = e_{j-i(mod 4)}$.

Example 6.2 Let R be a commutative ring and put $S = \sum_{1\leq i\leq 4} \oplus Re_i$, where $\{e_1, e_2, e_3, e_4\}$ is a set of non-zero orthogonal idempotents whose sum is one. Denote by G the cyclic group generated by σ of order 5. The mappings defined on the ideals $\{S, S_\sigma = Re_2, S_{\sigma^2} = Re_4, S_{\sigma^3} = Re_3, S_{\sigma^4} = Re_1\}$ by $\alpha_1 = id_S$, $\alpha_\sigma(e_1) = e_2$, $\alpha_{\sigma^2}(e_3) = e_4$, $\alpha_{\sigma^3}(e_4) = e_3$ and $\alpha_{\sigma^4}(e_2) = e_1$, give a partial action of G on S.

It can easily be verified that $S^G = R(e_1 + e_2) \oplus R(e_3 + e_4)$ and taking $x_i = y_i = e_i$, $1 \leq i \leq 4$, we have a Galois coordinate system for S over S^G. Thus S is an α-partial Galois extension of S^G. Also, it is not difficult to show that the enveloping action is given by $S' = S \oplus \sum_{1\leq j\leq 6} \oplus Rv_j$, where the set

$\{v_j | 1 \leq j \leq 6\}$ is a set of orthogonal idempotents which are also orthogonal with the e_i's and such that $\sum_i e_i + \sum_j v_j = 1$. The action of σ is given by $e_1 \to e_2 \to v_1 \to v_2 \to v_3 \to e_1$ and $e_3 \to v_4 \to e_4 \to v_5 \to v_6 \to e_3$. Here we have $S'^G = R(e_1 + e_2 + v_1 + v_2 + v_3) \oplus R(e_3 + e_4 + v_4 + v_5 + v_6)$.

Example 6.3 Let A be a cyclic (global) Galois extension of a commutative ring R with Galois group G generated by σ of order 6. Set $S = \sum_{1 \leq i \leq 5} \oplus Ae_i$, where $\{e_i \,|\, 1 \leq i \leq 5\}$ is a set of non-zero orthogonal idempotents of sum one. Define the partial action α of G on S taking $A_{\sigma^i} = A_{6-i}$ and $\alpha_{\sigma^i}(ae_i) = \sigma^i(a)e_{6-i}$, $1 \leq i \leq 5$. Thus we have a partial action of G on S and $S^G = \{ae_1 + be_2 + ce_3 + \sigma^2(b)e_4 + \sigma(a)e_5 \,|\, a, b \in A, c \in A^{\sigma^3}\}$.

Let $a_i, b_i \in A, 1 \leq i \leq m$, a Galois coordinate system for A over R and consider the elements $x_j = y_j e_j, j = 1, 2, 4, 5$ together with the elements $x_{i3} = a_i e_3, y_{i3} = b_i e_3$. It is easy to see that this gives a Galois coordinate system for S over S^G. Hence S is an α-partial Galois extension of S^G.

We can verify that the non-trivial separable α-strong subalgebras of S are $S_1 = \{x_1e_1 + x_2e_2 + x_3e_3 + \sigma^2(x_2)e_4 + x_5e_5 \,|\, x_i \in A\}$ and $S_2 = \{x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 \,|\, x_3 \in A^{\sigma^3}, x_i \in A \text{ for } i \neq 3\}$.

The above examples shows that is quite easy to construct partial Galois extensions. In certain sense, partial Galois extensions are ubiquitous:

Example 6.4 Assume that S is a Galois extension of R with Galois group G and A is an ideal of S. By the results in Section 4 of [5], if for any $\sigma \in G$ we define $A_\sigma = A \cap \sigma(A)$ and denote by $\alpha_\sigma : A_{\sigma^{-1}} \to A_\sigma$ the restriction of the automorphism σ of G, then we have a partial action of G on A. Under the additional assumption that A is generated by an idempotent element 1_A of A then, by the results of Section 4, S is an α-partial Galois extension of $R1_A$.

We can obtain a consequence of the main results in the case S is a reduced ring. Assume that S is reduced and denote by Q the complete ring of quotients os S. Recall that Q can be obtained considering the filter $\mathcal{F}$ of essential ideals of S and the set of all the S-homomorphisms $f : H \to S$, where $H \in \mathcal{F}$. Two homomorphism $f : H \to R$ and $f' : H' \to R$ are said to be equivalent if the restrictions of f and f' to $H \cap H'$ are equal. Then Q is defined as the set of all the equivalence classes of this homomorphisms with natural operations. Also, S can be considered as a subring of Q.

Let α be a partial action of the group G on S and we denote, as in the former sections, by $(S_\sigma)_{\sigma\in G}$ the ideals involved in the action. It is well-known that there is a one-to-one correspondence, via contraction, between closed ideals of S and closed ideals of Q. So for any $\sigma \in G$ there exists a closed ideal S^*_σ of Q such that $S^*_\sigma \cap S = [S_\sigma]$, where $[S_\sigma]$ denotes the closure of S_σ in S. Also, S^*_σ is the complete ring of quotients of the reduced ring (without identity) S_σ. Thus the isomorphisms $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma$ can be extended to isomorphisms $\alpha^*_\sigma : S^*_{\sigma^{-1}} \to S^*_\sigma$, which we will denote by α_σ again, and it can easily be seen that this defines a partial action α of G on Q. Note that this extended action satisfies the assumption we used in the former sections, i.e., the ideals S^*_σ have identity elements. For more details on this results the reader can see ([13], Section 1).

Let S be a reduced ring and α is a partial action of G on S. Denote by Q the complete ring of quotients of S and again by α the extended partial action of α to Q. As a consequence of Corollary 5.4 we obtain the following

Corollary 6.5 *Under the above notation, assume that there exist elements $x_i, y_i \in S$, $1 \le i \le n$, such that $b = \sum_{1\le i\le n} x_i y_i$ is a non-zero divisor of S and $\sum_{1\le i\le n} x_i \alpha_\sigma(y_i a) = 0$, for any $a \in S_{\sigma^{-1}}$ and $id \neq \sigma \in G$. Then Q is an α-partial Galois extension of Q^G and there exists a one-to-one correspondence between the subgroups of G and the Q^G-separable subalgebras of Q which are α-strong.*

Proof. As b is a non-zero divisor, then bS is an essential ideal of S and it follows that b is invertible in Q. Also, since S_σ is essential in S^*_σ, for any $\sigma \in G$, it easily follows that the elements elements $\{b^{-1}x_i, y_i, 1 \le i \le n\}$ give a Galois coordinate system in Q. Consequently Q is an α-partial Galois extension of Q^G and the result follows. □

Example 6.4 suggests that we could also consider the case in which the ideals S_σ are not necessarily generated by idempotents of S. But is not an easy problem to give a "good" definition of a partial Galois extension in this general case and to obtain some deep result, like the Fundamental Galois Theorem. This seems to be indicated by the fact that until now there is no a Galois Theory for rings without identity element. We were unable to obtain intrinsic results for this more general case in which there are not identities in the ideals S_σ, even when S is a reduced ring.

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